A Holomorphic 0-Surgery Model for Open Books with Application to Cylindrical Contact Homology

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Abstract

We give a simple model in $\mathbb{C}^2$ of the 0-surgery along a fibered knot of a closed 3-manifold M to yield a mapping torus $\hat{M}$. This model allows explicit relations between pseudoholomorphic curves in $\mathbb{R} \times M$ and in $\mathbb{R} \times \hat{M}$. We then use it to compute the cylindrical contact homology of open books resulting from a positive Dehn twist on a torus with boundary.

1 Introduction

In this note all 3-manifolds are closed and orientable, all contact structures are coorientable and all surfaces are oriented. Given a surface Σ a diffeomorphism $\phi \in \mathrm{Diff}^+(\Sigma)$ we denote by

$$\Sigma_\phi := \frac{\Sigma \times [0,1]}{(\phi(x), 0) \sim (x, 1)}$$

the mapping torus associated to ϕ. Throughout this paper we assume that a symplectic structure ω on Σ is given, $\phi \in \mathrm{Symp}(\Sigma, \omega)$ and $\phi = id$ near $\partial\Sigma$.

It is a basic fact in topology that a canonical 0-surgery along (every connected component of) the binding of an open book M yields a mapping torus $\hat{M}$. Now with the correspondence between contact structures and open books established by Thurston and Winkelnkemper [22] and Giroux [10, 11], it is expected that a "nice" description of the said 0-surgery will benefit the study of contact manifolds and symplectic manifolds.

For example, Eliashberg [7] showed that this surgery can be done *symplectically*, namely the two manifolds can be included as the boundary of a symplectic cobordism of which the symplectic structure satisfies some

boundary conditions pertaining to the given open book and mapping torus. His result leads to the equivalence between the (weakly) semi-symplectic fillability and the (weakly) symplectic fillability of contact manifolds, and provides applications to Kronheimer and Mrowka's Property P as well as Ozsváth and Szabó's Heegaard Floer homology theory. See also [5, 16].

Now, both M and $\hat{M}$ have a natural *symplectization* $\mathbb{R} \times M$ and $\mathbb{R} \times \hat{M}$ on which one can define holomorphic curve invariants with similar setups. On $\mathbb{R} \times M$ we have *contact homology* first constructed by Eliashberg and Hofer [6, 8] to provide Gromov-Floer type invariants for contact manifolds. On $\mathbb{R} \times \hat{M}$ there is Hutchings and Sullivan's *periodic Floer homology* for symplectic maps [15], which is a generalization of Seidel's *symplectic Floer homology* [20].

Here we are interested in the contact homology of contact 3-manifolds, which is in general very difficult to compute. To provide an access for computing and studying contact homology, it is then desirable to find a model demonstrating that the 0-surgery can be done *holomorphically* – at least in some reasonable sense, allowing an explicit correspondence between moduli of pseudoholomorphic curves and hence, a comparison of pseudoholomorphic curve theories on the two symplectic manifolds.

Indeed, such a model does exist and is very simple. Assume the binding B of M is connected for simplicity. Let $\mathcal{N}_B$ denote a small tubular neighborhood of B. Let $\hat{\mathcal{N}}_B := \hat{M} \setminus (M \setminus \mathcal{N}_B)$. $\hat{\mathcal{N}}_B$ is a tubular neighborhood of an orbit $\hat{e}$ corresponding to an elliptic fixed point of the monodromy of the mapping torus $\hat{M}$.

Let $\mathbb{C}^2$ be the standard complex plane. We will prove the following

Theorem 1.1. *There are simultaneous holomorphic embeddings of symplectizations $\mathbb{R} \times \mathcal{N}_B$ and $\mathbb{R} \times \hat{\mathcal{N}}_B$ into $\mathbb{C}^2$ with the intersection of the images an open domain in $\{z_1 z_2 \neq 0\}$*

In particular, with appropriate almost complex structures on $\mathbb{R} \times M$ and $\mathbb{R} \times \hat{M}$ given, Theorem 1.1 implies

Lemma 1.1. *Let $C \subset \mathbb{R} \times M$ be a pseudoholomorphic cylinder bounding Reeb orbits $\gamma_\pm$ at $\pm\infty$, with $\gamma_\pm$ not equal to any multiple of B. Suppose that C intersects with $\mathbb{R} \times B$ at s points with intersection multiplicities $m_1, ..., m_s \in \mathbb{N}$, then C lifts, via the canonical 0-surgery along B, to a $(2+s)$-punctured pseudoholomorphic sphere in $\mathbb{R} \times \hat{M}$. Moreover, the extra i^{th} puncture converge to the m_i^{th} iterate of $\hat{e}$ at $-\infty$.*

Let (M, ξ) be the contact 3-manifold associated to the open book (Σ, ϕ), where Σ is a punctured torus and the monodromy ϕ is a positive σ-Dehn

twist along an embedded nonseparating circle of Σ. We have $H_1(M,\mathbb{Z}) = \mathbb{Z}_\sigma \oplus \mathbb{Z}$. The holomorphic 0-surgery model enables us to relate certain holomorphic cylinders in $\mathbb{R} \times M$ to Taubes's trice-punctured spheres [21] (see also [15]), and by using Bourgeois's Morse-Bott version of contact homology We obtain the following

Theorem 1.2. *The cylindrical contact homology $HC(M, \xi, \mathbb{Q})$ is freely generated by*

1. h^m *(hyperbolic) with* $m \in \mathbb{N}$, $[h^m] = 0 \in H_1(M,\mathbb{Z})$, *and by*
2. $E_{i,m}$ *(elliptic) with* $(i,m) \in \mathbb{Z}_\sigma \times \mathbb{N} \backslash \{(0,1)\}$, $[E_{i,m}] = (i,0) \in H_1(M,\mathbb{Z})$.

Moreover, $[h^m] = [E_{0,m}] = 0 \in H_1(M,\mathbb{Z})$, *and their reduced Conley-Zehnder indexes are*

$$\bar{\mu}(h^m) = 2m - 1 \quad m \in \mathbb{N}; \quad \bar{\mu}(E_{0,m}) = 2m - 2, \quad m \in \mathbb{N}_{\geq 2}.$$

Readers are referred to Proposition 4.2 and Definition 4.2 for the definitions of h^m and $E_{i,m}$.

Note that when the positive Dehn twist is simple ($\sigma = 1$), the open book is $S^1 \times S^2$ with the unique (up to isotopy) Stein-fillable contact structure, of which the cylindrical contact has been computed in [24] via *subcritical* contact handle attaching. However, for $\sigma \geq 2$ the contact manifold is Stein-fillable but *not* subcritical Stein-fillable, nor an S^1-bundle over a closed surface (the contact homology of S^1-bundles have been computed, see [8][1]). Our results here provide first examples of cylindrical contact homology via nontrivial (i.e. monodromy $\neq id$) open books. We hope that the holomorphic 0-surgery model will lead to more examples of (cylindrical) contact homology of open books as well as new results in contact topology.

This paper is organized as follows. Section 2 consists of some background on cylindrical contact homology, contact structures associated to open books, 0-surgery and mapping tori. In Section 3 we construct in $\mathbb{C}^2$ a holomorphic model of a 0-surgery and verify Theorem 1.1 and Lemma 1.1. The computation of the cylindrical contact homology of a positive Dehn twist is done in Section 4.

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2 Background

2.1 Cylindrical contact homology

Contact forms. A 1-form $\alpha \in \Omega^1(M)$ $(\dim M = 3)$ is said to be *contact* if $\alpha \wedge d\alpha$ is nowhere vanishing. The kernel $\xi := \ker \alpha$ is called a *contact structure*. We say α and hence ξ are *positive* if $\alpha \wedge d\alpha$ is a volume form of the oriented manifold M. In this paper all contact 1-forms considered are *positive*.

Reeb orbits. There associates to α a unique vector field $R = R_\alpha$ called *Reeb* vector field, which is defined by

$$d\alpha(R, \cdot) = 0, \quad \alpha(R) = 1.$$

A periodic integral trajectory of R is called a *Reeb orbit* (of α). We call γ *simple* if γ is not a nontrivial multiple cover of another Reeb orbit.

Notation 2.1. We denote by γ^m the m^{th}-iterate of a Reeb orbit γ.

Definition 2.1 (action). Let $\gamma : [o, \tau] \to M$ be a Reeb trajectory with $\dot\gamma(t) = R_\alpha(\gamma(t))$. Define the *action* $\mathcal{A}(\gamma)$ of γ to be the number

$$T = \mathcal{A}(\gamma) := \int_\gamma \alpha \tag{1}$$

The flow R^t of R preserves ξ. Thus the linearized Reeb flow R^t_* , when restricted on γ, defines a path of symplectic maps

$$\Lambda_\gamma(t) = R^t_*(\gamma(0)) : \xi|_{\gamma(0)} \to \xi|_{\gamma(t)} \ .$$

Linearized Poincaré return map. When γ is a Reeb orbit with action T, $\Lambda_\gamma := \Lambda_\gamma(T)$ is called the *linearized Poincaré return map* along γ.

Definition 2.2. A Reeb orbit γ is *non-degenerate* if 1 is not an eigenvalue of its linearized Poincaré return map Λ_γ. A contact 1-form α is called *regular* if every Reeb orbit of α is non-degenerate.

It is well-known that generic contact 1-forms are regular (see [2]).

Definition 2.3 (good orbit). A Reeb orbit is said to be *bad* (see Section 1.2 of [8]) if it is an even multiple of another Reeb orbit whose linearized Poincaré return map has the property that the total multiplicity of its eigenvalues from the interval $(-1, 0)$ is odd. A Reeb orbit is *good* if it is not bad.

Notation 2.2. We denote by $\mathcal{P}_\alpha$ the set of all *good* Reeb orbits of α.

A mod 2 index. Assume that α is regular. Then for *any* Reeb orbit γ, a $\mathbb{Z}_2$-index $\bar{\mu}(\gamma, \mathbb{Z}_2)$ is defined:

$$\bar{\mu}(\gamma; \mathbb{Z}_2) = \begin{cases} 0 & \text{if } \gamma \text{ is } \textit{even}, \text{ i.e., if } \det(\Lambda_\gamma - Id) > 0; \\ 1 & \text{if } \gamma \text{ is } \textit{odd}, \text{ i.e., if } \det(\Lambda_\gamma - Id) < 0. \end{cases} \tag{2}$$

$\bar{\mu}$-index. Since $d\alpha|_\xi$ is a symplectic 2-form, the first Chern class $c_1(\xi) \in H^2(M, \mathbb{Z})$ is defined. For the sake of simplicity we assume in this paper that $c_1(\xi) = 0$ on $H_2(M, \mathbb{Z})$. Then the Conley-Zehnder index $\mu(\gamma)$ of a *homologously trivial* Reeb orbit γ is well-defined ([19][18]).

Definition 2.4. The *reduced* Conley-Zehnder index is defined to be

$$\bar{\mu}(\gamma) := \mu(\gamma) - 1 \quad \text{ if } \dim M = 3. \tag{3}$$

When γ is homologously trivial,

$$\bar{\mu}(\gamma) \equiv \bar{\mu}(\gamma, \mathbb{Z}_2) \qquad \text{mod } 2. \tag{4}$$

Almost complex structures. Following Eliashberg, Givental and Hofer [6][8] one can define Gormov-Floer type invariant called *contact homology* for (M, ξ) by counting in the symplectic manifold (called the *symplectization* of (M, α)) $(\mathbb{R} \times M, d(e^t\alpha))$ pseudoholomorphic curves bounding Reeb orbits at $\pm\infty$. The almost complex structure J involved is α-*admissible*, i.e.,

1. J preserves ξ;
2. $J|_\xi$ is $d\alpha$-compatible, i.e., $d\alpha(v, Jv) > 0$ for all $0 \neq v \in \xi$ and $d\alpha(v, w) = d\alpha(Jv, Jw)$ for all $v, w \in \xi$;
3. $J(\partial_t) = R_\alpha$, $J(R_\alpha) = -\partial_t$.

Pseudoholomorphic cylinders and planes. Fix a pair (α, J) with α regular and J an α-admissible almost complex structure on $\mathbb{R} \times M$. Given two good Reeb orbits γ_- and γ_+ we denote by $\mathcal{M}(\gamma_-, \gamma_+)$ the moduli space of maps $(\tilde{u}, j)$ where

1. j is an almost complex structure on S^2 (here we identify S^2;
2. let $\dot{S}^2 := S^2 \setminus \{0, \infty\}$, then $\tilde{u} = (a, u) : (\dot{S}^2, j) \to (\mathbb{R} \times M, J)$ is a proper map and is (j, J)-holomorphic, i.e., $\tilde{u}$ satisfies $d\tilde{u} \circ j = J \circ d\tilde{u}$;
3. $\tilde{u}$ is asymptotically cylindrical over γ_- at the negative end of $\mathbb{R} \times M$ at the puncture $0 \in S^2$; and $\tilde{u}$ is asymptotically cylindrical over γ_+ at the positive end of $\mathbb{R} \times M$ at the puncture $\infty \in S^2$;
4. $(\tilde{u}, j) \sim (\tilde{v}, j')$ if there is a diffeomorphism $f : \dot{S}^2 \to \dot{S}^2$ such that $\tilde{v} \circ f = \tilde{u}$, $f_* j = j'$, and f fixes all punctures.

For a *contractible* Reeb orbit γ The moduli space $\mathcal{M}(\gamma)$ of pseudoholomorphic planes bounding γ at ∞ is defined in a similar fashion.

For generic α-admissible J, $\mathcal{M}(\gamma_-, \gamma_+)$ and $\mathcal{M}(\gamma)$, if not empty, are smooth manifolds on which $\mathbb{R}$ acts freely by translation. If both $\gamma_\pm$ are homologously trivial then (see [8] Proposition 1.7.1)

$$\dim \mathcal{M}(\gamma_-, \gamma_+) = \bar{\mu}(\gamma_+) - \bar{\mu}(\gamma_-). \tag{5}$$

In particular, if $\dim \mathcal{M}(\gamma_-, \gamma_+) = 1$ then $\mathcal{M}(\gamma_-, \gamma_+)/\mathbb{R}$ is compact 0-dimensional, hence a finite number of points.

For contractible γ we have

$$\dim \mathcal{M}(\gamma) = \bar{\mu}(\gamma). \tag{6}$$

Energy. If $\tilde{u} = (a, u) \in \mathcal{M}(\gamma_-, \gamma_+)$ (or $\mathcal{M}(\gamma)$) then $u^* d\alpha \geq 0$ pointwise, and vanishes at most at finitely many points. We define the *contact energy* $E(\tilde{u})$ of $\tilde{u}$ to be

$$E(\tilde{u}) := \int_{u(\dot{S}^2)} d\alpha = \int_{\gamma_+} \alpha - \int_{\gamma_-} \alpha = \mathcal{A}(\gamma_+) - \mathcal{A}(\gamma_-) \geq 0.$$

Note that $E(\tilde{u}) = 0$ iff $\gamma_- = \gamma_+$, and in this case the moduli space consists of a single element $\mathbb{R} \times \gamma_+$.

For $\tilde{u} = (a, u) \in \mathcal{M}(\gamma)$ the contact energy is defined similarly:

$$E(\tilde{u}) := \int_{u(\mathbb{C})} d\alpha = \int_{\gamma} \alpha = \mathcal{A}(\gamma) > 0.$$

Contact complex. The contact complex $\mathcal{C}(\alpha)$ is the free module over $\mathbb{Q}$ generated by all elements of $\mathcal{P}_\alpha$ the set of all *good* Reeb orbits.

Boundary operator ∂. For a Reeb orbit γ we denote by κ_γ its multiplicity. Similarly we denote by κ_C the multiplicity of a pseudoholomorphic curve C in $\mathbb{R} \times M$.

The *boundary operator* ∂ of the contact complex $\mathcal{C}(\alpha)$ is defined by (see [7][2] but for a different coefficient ring)

$$\partial\gamma := \sum_{\gamma' \in \mathcal{P}_\alpha} \langle \partial\gamma, \gamma' \rangle \gamma' \tag{7}$$

$$\langle \partial\gamma, \gamma' \rangle := \kappa_\gamma \sum_{\substack{C \in \mathcal{M}(\gamma',\gamma)/\mathbb{R} \\ \dim \mathcal{M}(\gamma',\gamma)=1}} \frac{\pm 1}{\kappa_C} \tag{8}$$

The $\pm$ sign in (8) depends on the orientation of $C \in \mathcal{M}(\gamma', \gamma)/\mathbb{R}$ (see Section 4.5).

Definition 2.5. Suppose that $\partial^2 = 0$. Then the *cylindrical contact homology* of (M, ξ, α, J) is defined to be $HC(M, \xi, \alpha, J) := \ker \partial / \mathrm{im}\partial$.

Remark 2.1. In contact homology [8] one defines the boundary operator $d = \sum_{i=0}^\infty d_i$ by counting 1-dimensional moduli of holomorphic spheres with one positive puncture and arbitrary number of negative punctures. The summand d_i counts the number of holomorphic spheres with i negative punctures. In particular $d_1 = \partial$. Since $d^2 = 0$ (see [8]) we have $d_1^2 + d_0 d_2 = 0$. Thus $\partial^2 = 0$ if $d_0 = 0$, which is the case for the examples that we will compute in Section 4.

Contractible subcomplex. Let $\mathcal{C}^o(\alpha)$ denote the subcomplex generated by all good *contractible* Reeb orbits. Recall that we assume $c_1(\xi) = 0$ on $H_2(M, \mathbb{Z})$, hence $\bar{\mu}$-index is well-defined for all contractible Reeb orbits. Thus $\mathcal{C}^o(\alpha)$ is graded by $\bar{\mu}$. We denote by $\mathcal{C}^o_k(\alpha) \subset \mathcal{C}^o(\alpha)$ the subcomplex generated by all elements of $\mathcal{C}^o(\alpha)$ with $\bar{\mu} = k$.

Theorem 2.1 (see [23][8]). *Assume that $\partial^2 = 0$. Suppose that $\mathcal{C}^o_k(\alpha) = 0$ for $k = 0, -1$, then the cylindrical contact homology $HC(M, \xi) := HC(M, \xi, \alpha, J)$ is independent of the contact form α, the almost complex structure J; it depends only on the isotopy class of the contact structure ξ.*

2.2 Open book, contact structure and mapping torus

Open book. The pair (Σ, ϕ) is said to be an *open book* representation of a 3-manifold M if M can be expressed as

$$M = \Sigma_\phi \cup_{id} (B \times D^2)$$

where

- $\phi \in \mathrm{Diff}^+(\Sigma, \partial\Sigma)$, $\phi = id$ near $\partial\Sigma$ is the *monodromy*, and
- $B \cong \partial\Sigma$ is called the *binding* of the open book.

The complement $M \setminus B$ fibers over S^1, the fibers are called *pages* and are diffeomorphic to Σ.

Positive stabilization. An open book (Σ', ϕ') is called a *positive stabilization* of (Σ, ϕ) if

- Σ' is obtained by gluing to $\partial\Sigma$ the boundaries $\{\pm 1\} \times [-\epsilon, \epsilon]$ of a strip (a 2-dimensional handle of index 1) $[-1, 1] \times [-\epsilon, \epsilon]$ (so the Euler number of the page is decreased by 1, i.e., $\chi(\Sigma') = \chi(\Sigma) - 1$),
- $\phi' := \phi \circ \tau_\Gamma$; where $\Gamma \subset \Sigma'$ is an embedded circle intersecting with the cocore $\{0\} \times [-\epsilon, \epsilon]$ at a single point, and τ_Γ is the *positive* Dehn twist along Γ.

Two open books are said to be *equivalent up to positive stabilizations* if they become equal after applied with finitely many positive stabilizations.

With positive stabilizations we may assume that B *is connected* [10].

Associated contact structure. The work of Thurston and Winkelnkemper [22] and Giroux [10] established the following important bijection:

$$\frac{\text{contact structures on } M}{\text{contact isotopies}} \leftrightarrow \frac{\text{open books of } M}{\text{positive stabilizations}}.$$

Here we sketch the construction of a contact 1-form associated to an open book (see [10]). For an open book (Σ, ϕ) with a connected binding B, we fix an area form $\omega = d\beta$ on Σ. Isotope ϕ if necessary we may assume that $\phi^*\omega = \omega$. Then associate to (Σ, ϕ) a contact 1-form α such that

- $d\alpha$ restricts to an area form on every page, $d\alpha = \omega$ on each fiber of Σ_ϕ, and
- $\alpha = dp + r^2 dt$ near the binding B, where p parametrizes $B \cong S^1$, (r, t) are the polar coordinates of the D^2 factor of $B \times D^2$.

In particular, on Σ_ϕ we can define α to be

$$\alpha := (1 - t)\beta + t\phi^*\beta + K dt.$$

Then α is contact on Σ_ϕ provided that K is a large enough constant.

Remark 2.2. If α is associated to an open book (Σ, ϕ) then the Reeb orbits of α correspond to the the periodic points of some $\phi' \in \mathrm{Symp}(\Sigma, \omega)$ isotopic to ϕ, and positive multiples of B.

Mapping torus from 0-surgery. A canonical 0-surgery along the binding B of an open book (Σ, ϕ) of a 3-manifold M yields the mapping torus $\hat{M} = \hat{\Sigma}_{\hat{\phi}}$ together with a special section $\hat{e} = \{z\} \times S^1$ of the fibration

$$\hat{\Sigma} \to \hat{M} \xrightarrow{\hat{\pi}} S^1,$$

where

- $\hat{\Sigma} = \Sigma \cup D^2$ is a closed Riemann surface,
- $z \in D^2 \subset \hat{\Sigma}$, $\hat{M} \setminus \hat{e} \overset{\text{diffeo}}{\cong} \Sigma_\phi$,
- $\hat{\phi} \in \mathrm{Symp}(\hat{\Sigma}, \hat{\omega})$ satisfies $\hat{\omega}|_\Sigma = \omega$, $\hat{\phi}|_\Sigma = \phi$, $\hat{\phi}|_{D^2} = id$ (so $\hat{\phi}(z) = z$).

Remark 2.3. The point $z \in \mathrm{Fix}(\hat{\phi})$ and hence the section $\hat{e} = \{z\} \times S^1$ remember the 0-surgery, so that a canonical 0-surgery along $\hat{e}$ gives back the open book $(\Sigma, \phi) = M$. Note that if the fixed points z, z' of $\hat{\phi}$ are distinct, then in general the sections $\{z\} \times S^1$, $\{z'\} \times S^1$ may not be isotopic (as knots) in $\hat{M}$ – they may even represent different elements of $\pi_1(\hat{M})$, hence correspond to different open books. The marked point z is essential for recovering the original open book and hence contact structure.

Extension of $d\alpha$ over $\hat{M}$. The 2-form $\hat{\omega}$ canonically extends to a 2-form on $\hat{\Sigma}_{\hat{\phi}}$ which we still denote by $\hat{\omega}$. The differential $d\alpha|_{\Sigma_\phi}$ extends to a closed 2-form $\hat{\tau}$ on $\hat{M}$ such that $\hat{\tau}$ pulls back to $\hat{\omega}$ on each fiber of $\hat{\pi}$. Such $\hat{\tau}$ can be expressed as

$$\hat{\tau} = \hat{\omega} + \eta_t \wedge dt,$$

where $t \in S^1 = \mathbb{R}/\mathbb{Z}$ is the coordinate of the base S^1, $\eta = \eta_t$ is a family of closed 1-forms on $\hat{\Sigma}$ satisfying $\hat{\phi}^* \eta_0 = \eta_1$.

Orbits. The horizontal distribution $\ker \hat{\tau}$ is generated by the vector field

$$\hat{R} := \hat{\partial}_t + X_\eta,$$

with $\hat{\partial}_t \subset \ker \hat{\omega}$, $\hat{\pi}_*(\hat{\partial}_t) = \partial_t$, and $X_\eta \subset \ker \hat{\pi}_*$ is the t-dependent symplectic vector field defined by the equation

$$\hat{\omega}(X_\eta, \cdot) = \eta_t.$$

Remark 2.4. Similar to Remark 2.2, $\hat{R}$-orbits correspond to periodic points of some $\hat{\phi}' \in \mathrm{Symp}(\hat{M}, \hat{\omega})$ symplectically isotopic to $\hat{\phi}$.

Remark 2.5. The 2-form $\hat{\tau}$ is the canonical extension of $\hat{\omega} \in \Omega^2(\hat{\Sigma})$ over $\hat{\Sigma}_{\hat{\phi}'} \cong \hat{\Sigma}_{\hat{\phi}}$ corresponding to the monodromy $\hat{\phi}'$.

Periodic Floer homology. Based on the idea of Seidel [20], Hutchings and Sullivan [15] defined *Periodic Floer homology* for $\hat{\phi}' \in \mathrm{Symp}(\hat{\Sigma}, \hat{\omega})$ by counting in the symplectic manifold

$$(\mathbb{R} \times \hat{M}, \hat{\tau} + ds \wedge dt), \quad s \in \mathbb{R},$$

pseudoholomorphic curves converging to periodic trajectories of $\hat{R}$ at $s = \pm\infty$. The relevant almost complex structure $\hat{J}$ will be

1. $\mathbb{R}$-invariant,
2. *tamed* by $\hat{\Omega} := \hat{\tau} + ds \wedge dt$, i.e. $\hat{\Omega}(v, \hat{J}v) \neq 0$ for $v \neq 0$, and
3. $\hat{J}(\partial_s) = f\hat{R}$, $\hat{J}(f\hat{R}) = -\partial_s$ for some positive function f on $\hat{M}$.

A motivation. Now that each of (M, ξ) and $\hat{M}$ has its own theory based on counting holomorphic curves, and these two theories have some very similar ingredients, namely periodic trajectories and almost complex structures. In fact we can have

$$R = f\hat{R} \quad \text{and} \quad J = \hat{J} \quad \text{ on } \Sigma_\phi.$$

So if we would like to compute (cylindrical) contact homology of (M, ξ), and since in doing so we need to know how to count holomorphic curves converging to B^m or intersecting with $\mathbb{R} \times B$, it helps to know how such curves correspond to $\hat{J}$-holomorphic curves in $\mathbb{R} \times \hat{M}$, if such a correspondence does exist. Moreover, One can use the correspondence to compare the two holomorphic curve theories, exploring the relation between contact homology theory and mapping class groups.

Our goal in the next section is to establish a holomorphic 0-surgery model that allows a direct comparison of holomorphic curves before and after the surgery.

3 A holomorphic 0-surgery model

In this section we construct a holomorphic model of the canonical 0-surgery along the binding of an open book. It is no news that one can describe a 0-surgery in $\mathbb{C}^2$. The novelty here is to do it carefully enough and to find a nice vector field Y whose flow (i) preserves the standard complex structure J_o on $\mathbb{C}^2$, and (ii) embeds the two symplectizations into $\mathbb{C}^2$ nicely so that J_o is their common almost complex structure (see Lemma 3.3). This is done in Sections 3.1-3.3. Lemma 3.4 and 3.5 in Section 3.4 describe how punctured holomorphic discs in $\mathbb{C}^2$ are perceived in each symplectizations. Corollary 3.2 in particular will be used to determine the boundary operator (see Section 4.3). In Section 3.5 it is confirmed that correspondences between holomorphic curves in the local model extend straight forwardly to correspondence between pseudoholomorphic curves in $\mathbb{R} \times M$ and $\mathbb{R} \times \hat{M}$.

3.1 A contact solid torus in $\mathbb{C}^2$

Let $\mathbb{C}^2 := \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{C}\}$ be the complex plane with the standard complex structure J_o. Write $z_j = r_j e^{i\theta_j}$.

Consider on $\mathbb{C}^2$ the smooth function

$$F(z_1, z_2) := -|z_1|^2 + |z_2|^2 = -r_1^2 + r_2^2.$$

Let $\epsilon > 0$ be a constant and define

$$N = N_\epsilon := F^{-1}(-1) \cap \{r_2 < \epsilon\}. \tag{9}$$

N is diffeomorphic to $S^1 \times D^2$. Let $\iota : N \hookrightarrow \mathbb{C}^2$ denote the inclusion map. Define

$$\lambda := \iota^*(\frac{-1}{2} dF \circ J_o) = -(1 + r_2^2) d\theta_1 + r_2^2 d\theta_2.$$

Fact 3.1. *The 1-form λ is a contact on N, its contact structure and Reeb vector field are*

$$\zeta := \ker \lambda = TN \cap J_o TN, \qquad R_\lambda = -\partial_{\theta_1} - \partial_{\theta_2}$$

Then ζ is the maximal complex subbundle of the tangent bundle of N. (N, ζ) will be a model of a tubular neighborhood of the binding. Note that all trajectories of R_λ are periodic. We would like to perturb λ, but keeping ζ intact, so that the resulting Reeb vector field has only one simple periodic orbit namely, $\gamma := \{r_1 = 1\} \cap N$ with orientation given by $-\partial_{\theta_1}$.

Lemma 3.1. *For any constant $c > 0$ there is a smooth function $h = h(r_2) \in C^\infty(N)$ depending only on r_2 and c such the the Reeb vector field $R_{\lambda'}$ of the contact 1-form $\lambda' := e^{-h}\lambda$ is $R_{\lambda'} = -\partial_{\theta_1} + c\partial_{\theta_2}$. In particular, if $c \in \mathbb{R}_+ \setminus \mathbb{Q}$ then the only simple Reeb orbit of $R_{\lambda'}$ is $\gamma := \{r_1 = 1\} \cap N$ with orientation given by $-\partial_{\theta_1}$.*

Proof. Let $h = h(r_2) \in C^\infty(N)$ then the Reeb vector field of $\lambda' := e^{-h}\lambda$ is

$$R_{\lambda'} = e^h(R_\lambda + Z_h),$$

where $Z_h \subset \zeta$ is the unique vector field satisfying

$$d\lambda(Z_h, \cdot) = -dh \quad \text{ on } \zeta.$$

A straightforward calculation yields

$$Z_h = \frac{h'}{2r_2}(r_2^2\partial_{\theta_1} + (1 + r_2^2)\partial_{\theta_2}), \qquad\qquad (h' := \frac{dh}{dr_2}).$$

Then

$$R_{\lambda'} = e^h\Big(\big(\frac{r_2 h'}{2} - 1\big)\partial_{\theta_1} + \big(\frac{(1 + r_2^2)h'}{2r_2} - 1\big)\partial_{\theta_2}\Big).$$

Fix a positive constant c and solve for h satisfying the initial value problem

$$\frac{\frac{(1+r_2^2)h'}{2r_2} - 1}{\frac{r_2 h'}{2} - 1} = -c, \quad h(0) = 0$$

We get

$$h(r_2) = \ln(1 + (1 + c)r_2^2),$$

and

$$R_{\lambda'} = \iota^*(X), \quad X := -\partial_{\theta_1} + c\partial_{\theta_2}. \tag{10}$$

□

Let

$$\pi_N : N \setminus \gamma \to S^1_{\theta_2}$$

denote the projection onto the θ_2-coordinate. Then (π_N, γ) is an open book representation of N with pages diffeomorphic to an annulus. The monodromy ψ, which is the time 1 map of the flow of $\frac{1}{c}R_{\lambda'}$, is isotopic to the identity map.

Corollary 3.1. *The 2-form $d\lambda'$ is θ_2-independent and is symplectic when restricted to any page of π_N. Let ω denote the restriction of $d\lambda'$ to a page. Then $\psi^*\omega = \omega$.*

The following lemma shows that the binding γ and its positive iterates γ^m are *elliptic* (see (2)). This result will be used in Section 4.2.

Lemma 3.2. *Let $R_{\lambda'}$ be as in (10) with $0 < c \notin \mathbb{Q}$. Let $\gamma := N \cap \{r_1 = 1\}$ be the unique simple Reeb orbit on N. Then for $m \in \mathbb{N}$, γ^m is elliptic, i.e., $\bar{\mu}(\gamma^m, \mathbb{Z}_2) = 0$.*

Proof. Note that

$$R_{\lambda'} = x_1 dy_1 - y_1 dx_1 + c(x_2 dy_2 - y_2 dx_2).$$

Let $\zeta := \ker \lambda'$. Since $\zeta|_\gamma = \operatorname{span}(\partial_{x_2}, \partial_{y_2})$, then with respect to the ordered basis $\{\partial_{x_2}, \partial_{y_2}\}$,

$$DR_{\lambda'}|_\gamma = \begin{bmatrix} 0 & -2c \\ 2c & 0 \end{bmatrix}.$$

Since the action of γ^m is $2m\pi$, the linearized Poincaré return map of the flow of $R_{\lambda'}$ along γ^m is

$$\Lambda_{\gamma^m} = \exp\left(2m\pi \begin{bmatrix} 0 & -2c \\ 2c & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos 4mc\pi & -\sin 4mc\pi \\ \sin 4mc\pi & \cos 4mc\pi \end{bmatrix}.$$

If $c \notin \mathbb{Q}$ then $\det(\Lambda_{\gamma^m} - Id) < 0$, so γ^m is elliptic. $\square$

3.2 A 0-surgery in $\mathbb{C}^2$

Recall the small constant $\epsilon > 0$ and

$$N = N_\epsilon := F^{-1}(-1) \cap \{r_2 < \epsilon\}.$$

On N we apply the canonical 0-surgery along γ to get a new manifold

$$\hat{N} := \{-e^{-2s} r_1^2 + e^{2cs} r_2^2 = 1\} \cap \{r_2 < \epsilon\}$$

satisfying the following conditions:

1. $s = s(r_2) > 0$ is a smooth function depending only on r_2,
2. $\hat{N} \pitchfork Y$ where $Y := -J_o X = -r_1 \partial_{r_1} + c r_2 \partial_{r_2}$,
3. $\hat{N} \cap \{r_2 > \epsilon'\} = N \cap \{r_2 > \epsilon'\}$ for some positive constant $\epsilon' < \epsilon$.

Note that $\hat{N}$ is a solid torus diffeomorphic to $D^2 \times S^1_{\theta_2}$. Let

$$\pi_{\hat{N}} : \hat{N} \to S^1_{\theta_2}$$

denote the projection onto the θ_2-coordinate. $\pi_{\hat{N}}$ gives $\hat{N}$ the structure of a disc bundle over $S^1_{\theta_2}$.

The vector field $X = -\partial_{\theta_1} + c\partial_{\theta_2}$ is tangent to $\hat{N}$ and transversal to the fibers of $\pi_{\hat{N}}$. Together $\pi_{\hat{N}}$ and X induce on $\hat{N}$ the structure of a mapping torus of a disc with monodromy $\hat{\psi}$ induced by the time 1 map of the flow of $\frac{1}{c}X$.

Let $\Omega := r_1 dr_1 \wedge d\theta_1 + r_2 dr_2 \wedge d\theta_2$. Ω is the standard symplectic 2-form on $\mathbb{C}^2$. Ω is invariant under the flow of X and pulls back to a symplectic 2-form on every page of $\pi_{\hat{N}}$ (as well as on every fiber of $\pi_N : N \setminus \gamma \to S^1_{\theta_2}$). $\hat{\psi}$ is symplectic with respect to this pulled back 2-form, and is isotopic to the identity map. $\hat{\psi}$ has only one fixed point, which corresponds to the simple loop

$$\hat{\gamma} := \hat{N} \cap \{r_1 = 0\}, \quad \dot{\gamma} = \frac{1}{c} X.$$

3.3 Two overlapping symplectizations in $\mathbb{C}^2$

Recall the vector field

$$Y := -J_o X = -r_1 \partial_{r_1} + c r_2 \partial_{r_2}.$$

Let Y^t, $t \in \mathbb{R}$, denote the flow of Y.

Fact 3.2. *The flow Y^t preserves J_o and X.*

Note that Y^t also preserves the values $\rho := r_1^c r_2$, θ_1 and θ_2. The integral trajectories of Y^t on $\mathbb{C}^2 \setminus \{0\}$ are parametrized by $(\rho, \theta_1, \theta_2)$, $\rho > 0$, $\theta_1, \theta_2 \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

Recall from Lemma 3.1 the contact 1-form λ' on N. The contact structure $\zeta = \ker \lambda'$ is a J_o-complex line bundle, $J_o|_\zeta$ is $d\lambda'$-compatible, and $J_o Y|_N = X|_N$ is the Reeb vector field $R_{\lambda'}$.

With the above understood we can embed the symplectization $\mathbb{R} \times N$ into $\mathbb{C}^2$ by

$$\text{identifying } \{t\} \times N \text{ with } Y^t N, \quad \forall t \in \mathbb{R}, \tag{11}$$

where $Y^t N$ is the image of N under the time t map of the flow of Y. In particular the vector field ∂_t of $\mathbb{R} \times N_\epsilon$ is identified with Y. Let

$$W := \cup_{t \in \mathbb{R}} Y^t N \cong \mathbb{R} \times N. \tag{12}$$

Observe that J_o is Y-invariant hence is a λ'-admissible almost complex structure on $\mathbb{R} \times N \cong W$.

Likewise, we can also embed the "symplectization" $\mathbb{R} \times \hat{N}$ into $\mathbb{C}^2$ via the flow of Y by

$$\text{identifying } \{t\} \times \hat{N} \text{ with } Y^t\hat{N}, \quad \forall t \in \mathbb{R}. \tag{13}$$

Now the vector field ∂_t of $\mathbb{R} \times \hat{N}$ is identified with Y as well. Let

$$\hat{W} := \cup_{t\in\mathbb{R}} Y^t\hat{N} \cong \mathbb{R} \times \hat{N}. \tag{14}$$

Again, J_o is Y-invariant and Ω-compatible.

Lemma 3.3. *The symplectizations $\mathbb{R} \times N$ and $\mathbb{R} \times \hat{N}$ are overlapping on the region $W \cap \hat{W} \subset \mathbb{C}^2 \setminus \{z_1 z_2 = 0\}$, and they share the same compatible complex structure J_o. Hence a J_o-holomorphic map into $W \cap \hat{W}$ is a J_o-holomorphic map into both symplectizations.*

3.4 Holomorphic maps into $\mathbb{R} \times N$ and $\mathbb{R} \times \hat{N}$

Recall that the binding of N is the Reeb orbit

$$\gamma := N \cap \{r_2 = 0\} \text{ oriented by } X|_\gamma.$$

The 0-surgery along γ has the effect of replacing the open book N by the mapping torus $\hat{N}$, and the binding γ by the orbit

$$\hat{\gamma} := \hat{N} \cap \{|z_1| = 0\} \text{ oriented by } (X)|_{\hat{\gamma}}.$$

With the notations

$$\mathbb{C}^*_{z_1} := (\mathbb{C} \setminus \{0\}) \times \{0\}, \qquad \mathbb{C}^*_{z_2} := \{0\} \times (\mathbb{C} \setminus \{0\}),$$

we have

$$\mathbb{R} \times \gamma \cong \cup_{t\in\mathbb{R}} Y^t\gamma = \mathbb{C}^*_{z_1}, \tag{15}$$
$$\mathbb{R} \times \hat{\gamma} \cong \cup_{t\in\mathbb{R}} Y^t\hat{\gamma} = \mathbb{C}^*_{z_2}. \tag{16}$$

The union of both is then contained in the set

$$z_1 z_2 = 0.$$

Let $U \subset \mathbb{C}$ be an open disc containing the point $z = 0$. Denote $U^* := U \setminus \{0\}$. We are interested in J_o-holomorphic maps

$$f(z) = (f_1(z), f_2(z)) := U^* \to W \cap \hat{W} \subset \mathbb{C}^2 \setminus \{z_1 z_2 = 0\}$$

satisfying

Condition 3.1. *$f(z)$ converges to a positive multiple of either γ or $\hat{\gamma}$ at either $t = \infty$ or $t = -\infty$ as $z \to 0$.*

Recall that Y^t preserves the value $r_1^c r_2$, so two points $(z_1, z_2), (z_1', z_2')$ in $\mathbb{C}^2$ are projected by Y^t to the same point either in N or in $\hat{N}$ only if $|z_1|^c|z_2| = |z_1'|^c|z_2'|$. Since $\mathbb{R} \times \gamma \cup \mathbb{R} \times \hat{\gamma}$ is contained in the set $\{z_1 z_2 = 0\} = \{r_1^c r_2 = 0\}$, then a sequence of points $z_k = (z_{1k}, z_{2k})$ in N (resp. in $\hat{N}$) approach γ (resp. $\hat{\gamma}$) implies that $z_{1k}^c z_{2k} \to 0$ as $k \to \infty$.

Condition 3.1 then implies that

$$f_1(z)^c f_2(z) \to 0 \quad \text{as } z \to 0. \tag{17}$$

Hence for $j = 1, 2$, either f_j can be holomorphically extended over $z = 0$ or f_j has a pole of finite order at $z = 0$. Since

$$Y^t(z_1, z_2) = (e^{-t} z_1, e^{ct} z_2)$$

so if $|f_j| \in O(|z|^{n_j})$ near $z = 0$ for some $n_j \in \mathbb{Z}$, $j = 1, 2$, then

$$cn_1 + n_2 > 0. \tag{18}$$

Remark 3.1. *Often we will take $c \in \mathbb{R}_+ \setminus \mathbb{Q}$ to be arbitrarily large, then (18) eventually implies that $n_1 \geq 0$ (when $c \to \infty$).*

The sign of n_1 (resp. n_2) determines how f is perceived from the point of view of N (resp. $\hat{N}$).

Lemma 3.4. *View f as a J_o-holomorphic map into the symplectization $\mathbb{R} \times N$. Then we have the following conclusions depending on the sign of n_1.*

1. *$n_1 > 0$. $f(z)$ converges to γ^{n_1} at $t = \infty$ asymptotically as $z \to 0$.*
2. *$n_1 = 0$. f can be holomorphically extended over $z = 0$. $f(U)$ intersects transversally and positively with $\mathbb{C}^*_{z_1}$. The intersection multiplicity is n_2.*
3. *$n_1 < 0$. $f(z)$ converges to γ^{-n_1} at $t = -\infty$ asymptotically as $z \to 0$.*

By interchanging n_1 and n_2 we get similar conclusion from the perspective of $\hat{N}$.

Lemma 3.5. *View f as a J_o-holomorphic map into the symplectization $\mathbb{R} \times \hat{N}$. Then we have the following conclusions depending on the sign of n_2.*

1. $n_2 > 0$. *$f(z)$ converges to $\hat{\gamma}^{n_2}$ at $t = -\infty$ asymptotically as $z \to 0$.*

2. $n_2 = 0$. *f can be holomorphically extended over $z = 0$. $f(U)$ intersects transversally and positively with $\mathbb{C}^*_{z_2}$. The intersection multiplicity is n_1.*

3. $n_2 < 0$. *$f(z)$ converges to $\hat{\gamma}^{-n_2}$ at $t = \infty$ asymptotically as $z \to 0$.*

When computing contact homology of an open book, one needs to take into account holomorphic curves intersecting with the holomorphic cylinder $\mathbb{R} \times B$, where B is the binding of the open book. In our local model here, $\mathbb{R} \times B$ is identified with $R \times \gamma = \mathbb{C}^*_{z_1}$. The following corollary, as a special case of Lemma 1.1, states that, in our local model, how such curves correspond to curves in $\mathbb{R} \times \hat{N}$.

Corollary 3.2. *There is a one-one correspondence between pseudoholomorphic discs in $\mathbb{R} \times N$ that intersect with $\mathbb{R} \times \gamma$ at one point with winding number $n_2 > 0$ and pseudoholomorphic half-cylinders in $\mathbb{R} \times \hat{N}$ that converges to $\hat{\gamma}^{n_2}$ at $t = -\infty$.*

3.5 From local to global

Recall the mapping torus $M = \Sigma_\phi \cup_{id} B \times D^2$, where $B \times D^2$ is diffeomorphic to a solid torus $S^1 \times D^2$. Let p be the angular coordinate of $B \cong \mathbb{R}/2\pi\mathbb{Z}$ and (r, θ) the polar coordinates of $D^2 = \{r < \epsilon\}$.

Recall $N \subset \mathbb{C}^2$ from (9) as well as the contact 1-form λ' and its Reeb vector field $R_{\lambda'}$ from Section 3.1. The diffeomorphism $\Phi : B \times D^2 \to N$ defined by

$$\Phi(p, r, \theta) = (\sqrt{1 + r^2} e^{-ip}, re^{i\theta}) \tag{19}$$

induces on $B \times D^2$ the contact 1-form $\Phi^* \lambda'$ which can be extended over Σ_ϕ to be a contact 1-form α on M whose contact structure ξ is supported by the open book (Σ, ϕ).

On $B \times D^2$ we have $R_\alpha = \Phi^*(X|_N)$ hence $B^m = \Phi^{-1}(\gamma^m)$, $m \in \mathbb{N}$, are the only orbits of R_α on $B \times D^2$.

With $B \times D^2$ identified with N, the mapping torus $\hat{M}$ obtained by a 0-surgery along B can be identified with

$$\hat{M} = \Sigma_\phi \cup_{\Phi|_{B \times S^1}} \hat{N} = \hat{\Sigma}_{\hat{\phi}}$$

where $\hat{\Sigma} = \Sigma \cup D^2$ is the closed Riemann surface obtained by gluing along $\partial\Sigma \cong S^1$ a 2-disc with $\hat{\phi}|_\Sigma = \phi$ and $\hat{\phi} = id$ on $\hat{\Sigma} \setminus \Sigma$. The Reeb vector field R_α induces a vector field $\hat{R}$ on $\hat{M}$ such that $\hat{R}|_{\Sigma_\phi} = R_\alpha$ and $\hat{R}|_{\hat{N}} = X|_{\hat{N}}$.

With the diffeomorphism Φ, the common almost complex structure J_o of $\mathbb{R} \times N$ and $\mathbb{R} \times \hat{N}$ extends over Σ_ϕ as an α-admissible almost complex structure. The holomorphic curve correspondence between $\mathbb{R}\times N$ and $\mathbb{R}\times \hat{N}$ now extends over $\mathbb{R}\times M$ and $\mathbb{R}\times \hat{M}$ straightforwardly. In particular, Lemma 3.4 and Lemma 3.5 together imply Lemma 1.1.

4 Cylindrical contact homology of a Dehn twist

We define the Dehn twist in Section 4.1 and show in Section 4.2 that, on the corresponding open book M, the Reeb orbits come in three different types (Proposition 4.2). Holomorphic cylinders are studied in Section 4.3. It is shown that under various topological and dimensional constraints, the cylindrical contact homology is defined for M, and there are only a handful of types of holomorphic cylinders to be counted. Lemma 4.15 gives a complete classification (up to sign) of 1-dimensional moduli of holomorphic cylinders, either intersecting with the binding or not. An energy estimate is calculated in Section 4.4 to help understand the coherent orientation of the moduli and hence the signs involved in the boundary operator of the cylindrical contact homology $HC(M,\xi)$ (Section 4.5). We complete the computation of $HC(M,\xi)$ in Section 4.6, the result is summarized in Theorem 1.2.

4.1 Dehn twist

Let $\Sigma := T^2 \setminus D^2$ be a 2-torus with a disc removed. Let $\Gamma \subset \Sigma$ be an embedded nonseparating circle, $U \subset \overset{\circ}{U'}, U' \subset \Sigma$ be closed tubular neighborhoods of Γ. Let (q,p) be the coordinates of $U' \cong [q'_-, q'_+] \times S^1$, $U \cong [q_-, q_+] \times S^1$, $q'_- < q_- < q_+ < q'_+$, $S^1 \cong \mathbb{R}/\mathbb{Z}$, $\Gamma \cong \{q_o\} \times S^1$ for some $q_o \in (q_-, q_+)$. We also use coordinates (q,p) near $\partial\Sigma$ so that $\partial\Sigma = \{q = const\}$ and $dq \wedge dp$ is an area form near $\partial\Sigma$. We fix a symplectic 2-from ω on Σ so that

$$\omega = d(qdp) \quad \text{ on } U' \text{ and near } \partial\Sigma.$$

Fix a natural number $\sigma \in \mathbb{N}$ and let $\phi \in \mathrm{Symp}(\Sigma, \omega)$ denote a σ-Dehn twist supported on $U \supset \Gamma$ such that

$$\begin{aligned} \phi &= id && \text{on } \Sigma \setminus U, \\ \phi(q,p) &= (q, p - f(q)) && \text{on } U = [q^-, q^+] \times S^1, \end{aligned}$$

with $f : [q^-, q^+] \to [0, \sigma]$ a smooth surjective increasing function satisfying

$$f'(q) \geq 0, \quad f'(q^\pm) = 0.$$

The mapping torus U_ϕ can be described topologically as follows: Let $T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ be a torus parametrized by (p,t) and let $[p]$, $[t]$ be the corresponding generators of $\pi_1(T^2)$. Take two thickened tori $[q_-, q_o] \times T^2$ and $[q_o, q_+] \times T^2$ and glue them together by using the map $g_\tau : \{q_o\} \times T^2 \to \{q_o\} \times T^2$,

$$g_\tau(p,t) = (p, t + \sigma p).$$

Proposition 4.1. *Write $\partial U_\phi = T_- \cup T_+$, where $T_- := \{q_-\} \times S^1_p \times S^1_t$ and $T_+ := \{q_+\} \times S^1_p \times S^1_t$. Let $[p] = [\Gamma]$ and $[t]$ denote the generators of $T_\pm \cong S^1_p \times S^1_t$ with respect to the coordinates (p,t). Then as elements of $\pi_1(U_\phi)$ we have*

$$\pi_1(T_-) \ni [p]^n[t]^m = [p]^{n+\sigma m}[t]^m \in \pi_1(T_+). \tag{20}$$

Proof. Consider the homotopy $H : [q_-, q_+] \times S^1_\tau \to [q_-, q_+] \times S^1_p \times S^1_t$,

$$H(q,\tau) := (q, n\tau + m\tau f(q), m\tau).$$

$\square$

4.2 Contact 1-form and Reeb orbits

Let M denote the 3-manifold represented by the open book (Σ, ϕ) as defined in Section 4.1 and let B denote the binding. In this section we will construct an explicit contact 1-form α associated to the open book (Σ, ϕ) and classify its Reeb orbits.

Constructing α. Let $\beta \in \Omega^1(\Sigma)$ be a primitive of ω such that $\beta = q dp$ on U and near $\partial\Sigma$. We have $\phi^*\beta - \beta = -qf'(q)dq$. Define

$$\alpha := \beta - tqf'(q)dq + Kdt,$$

with $K \in C^\infty(\Sigma)$ to be specified in Condition 4.1 below. It is easy to see that α is contact if K is a positive constant large enough. Note that $\alpha = \beta + Kdt$ away from the Dehn twisted regions.

If K is a constant then the Reeb vector field R on the mapping torus Σ_ϕ is $R = (\partial_t - qf'\partial_p)/(K - q^2 f')$. On $U \times [0,1]$ the flow of R takes a point (q,p) at $t = 0$ to the point $(q, p - qf'(q))$ at $t = 1$, then ϕ identifies $(q, p - qf'(q))$ at $t = 1$ with the point

$$(q, p - qf'(q) - f(q)) = (q, p - (qf(q))') \quad \text{at } t = 0.$$

The coordinate q is fixed under the flow of R. When $(qf(q))' \in \mathbb{Q}$ the q-level set, which is a 2-torus, is fibrated by R-orbits; while there are no periodic Reeb trajectories if $(qf(q))' \notin \mathbb{Q}$. So we get infinitely many S^1-families of R-orbits on the mapping torus U_ϕ. Moreover $(\Sigma \setminus U)_\phi = (\Sigma \setminus U) \times S^1_t$ is a trivial S^1-bundle with Reeb orbits as fibers. Such a contact 1-form α is not regular. Recall that α is regular if the set of Reeb orbits is discrete, otherwise α is called not regular.

Instead of letting K be a constant we want to choose K so that the resulting Reeb vector field has no periodic trajectories in U_ϕ. Let $K = K(q)$ on U, then K is ϕ-invariant hence can be thought as a function on U_ϕ that depends only on q. The Reeb vector field is then $(\partial_t - (qf' + K_q)\partial_p)/(K - q(qf' + K_q))$. Level $q = const$ has R-orbits if and only if $qf' + K_q + f = K_q + (qf)' \in \mathbb{Q}$. So if $K_q = -(qf)' - \tilde{r}$ for some $\tilde{r} \notin \mathbb{Q}$ then there will be no R-orbits on U_ϕ. Such K is equal to $-q(f + \tilde{r}) + c_0$ for some constant c_0. To ensure α being contact we need $K - qK_q - q^2 f > 0$, which implies that $c_0 > 0$.

Now consider a smooth function $K \in C^\infty(\Sigma)$ satisfying the following

Condition 4.1. *1. On U, $K = K(q) = -q(f + \tilde{r}) + c_0$ for some constant $c_0 > 0$, $\tilde{r} \notin \mathbb{Q}$, $\tilde{r} \lesssim 0$, i.e., $\tilde{r} < 0$ and $\tilde{r} \sim 0$.*

2. *$K = K(q)$ on U'; $K_q > 0$ for $q \in (q'_-, q_-)$; $K_q < 0$ for $q \in (q_+, q'_+)$; and $K_{qq} > 0$ on $U' \setminus U$.*

3. *K is a Morse function on $\Sigma \setminus U$, with only one critical point x_h which is hyperbolic; $|dK| \sim 0$ on $\Sigma \setminus U'$.*

Moreover, we can extend K over $B \times D^2$ so that on $B \times D^2$ (see (19) and Lemma 3.1)

$$\alpha = q(r)dp + K(r)d\theta = \Phi^* \lambda',$$

which will ensure that the only Reeb orbits of α on $B \times D^2$ are B^m, $m \in \mathbb{N}$.

Let $R = R_\alpha$ denote the corresponding Reeb vector field, then

$$R = \begin{cases} (\partial_t + (f + \tilde{r})\partial_p)/(K - qK_q - q^2 f) & \text{on } U_\phi, \\ (\partial_t - K_q \partial_p)/(K - qK_q) & \text{on } (U' \setminus U)_\phi, \\ (\partial_t - X_K)/(K - \beta(X_K)) & \text{on } (\Sigma \setminus U')_\phi, \end{cases} \tag{21}$$

where X_K is the Hamiltonian vector field of K relative to the symplectic 2-form $\omega = d\beta$, i. e., $d\beta(X_K, \cdot) = -dK$.

Types of Reeb orbits. The following proposition describes the three types (Type $\mathfrak{T}$, $\mathfrak{C}$ and $\mathfrak{B}$) of Reeb orbits on (M, α):

Proposition 4.2. *There are three types of Reeb orbits described as follows:*

1. *Type $\mathfrak{T}$ orbits. These orbits come from the σ-Dehn twist and lie in $U' \setminus U$. They are parametrized by the set*

$$\mathcal{S} := \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid 0 < n < \sigma m\}. \tag{22}$$

   (See Remark 4.1 below for more discussion on the parametrization.) When K unperturbed, each of such pair (n, m) represents an S^1-family of (degenerate) Reeb orbits contained in the 2-torus $\{K_q = \frac{-n}{m}\} \cap U'$. A further perturbation of K will turn each S^1-family of orbits into a pair of orbits denoted by $h_{n/m}$ and $e_{n/m}$, $h_{n/m}$ is hyperbolic while $e_{n/m}$ is elliptic (see [15]).

2. *Type $\mathfrak{C}$ orbits. They are h^m, $m \in \mathbb{N}$, where $h = \{x_h\} \times S^1_t$ is the simple Reeb orbit corresponding to the unique critical point x_h of $K|_{\Sigma \setminus U}$, and x_h is hyperbolic.*

3. *Type $\mathfrak{B}$ orbits. They are B^m, $m \in \mathbb{N}$.*

Remark 4.1. Recall the number $\tilde{r} \lesssim 0$, $\tilde{r} \notin \mathbb{Q}$, from Condition 4.1. The $\mathfrak{T}$-orbits in the region (q_+, q'_+) are parametrized by the set

$$\mathcal{S}_+ := \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid \ n < (\sigma + \tilde{r})m\},$$

where $(n, m) \in H_1(T_+ = S^1_p \times S^1_t, \mathbb{Z})$ is indeed the homology class of the corresponding orbits. Similarly the $\mathfrak{T}$-orbits in the region (q'_-, q_-) are parametrized by

$$\mathcal{S}_- := \{(n, m) \in (-\mathbb{N}) \times \mathbb{N} \mid \tilde{r}m < n < 0\},$$

where $(n, m) \in H_1(T_- = S^1_p \times S^1_t, \mathbb{Z})$.

Recall from (4.1) that an element $(n, m) \in H_1(T_-, \mathbb{Z})$ gets identified with $(n + \sigma m, m) \in H_1(T_+, \mathbb{Z})$. Thus $\mathcal{S}_-$ is identified with

$$\{(n, m) \in \mathbb{N} \times \mathbb{N} \mid (\sigma + \tilde{r})m < n < \sigma m\},$$

where $(n, m) \in H_1(T_+ = S^1_p \times S^1_t, \mathbb{Z})$. As a result, $\mathfrak{T}$-orbits are parametrized by their corresponding homology classes in T_+, i.e., by the set $\mathcal{S}$ in (22).

In fact, as $\tilde{r}$ can be arbitrarily close to 0, when $\tilde{r} \to 0$, all $\mathfrak{T}$-orbits are "pushed into" the region $q_+ < q < q'_+$ and hence are parametrized by $\mathcal{S}$.

Remark 4.2. Indeed there are Reeb orbits other than the three types discussed above. These extra orbits lie in the region between x_h and the collar of B, and all wind around B with winding numbers $> c$ (recall the number c from Lemma 3.1), and are homotopic to B^m for some $m \in \mathbb{N}$. Since c can be made arbitrarily large, these "transient" orbits disappear as $c \to \infty$. Moreover, as these orbits are homologously trivial and $c_1(\xi) = 0$ (see Lemma 4.2 below), their $\bar{\mu}$-indexes are defined and will become arbitrarily large as $c \to \infty$, hence they have no contribution to the contact homology. The only Reeb orbits responsible for the contact homology are of Type $\mathfrak{T}$, $\mathfrak{C}$ and $\mathfrak{B}$.

Lemma 4.1. *All Reeb orbits are* good*.*

Proof. First of all, from Lemma 3.2 one sees that for all $m \in \mathbb{N}$, B^m is elliptic and its Poincaré return map has no real eigenvalues (assuming $c \notin \mathbb{Q}$). So B^m is always good.

Secondly, h^m is hyperbolic for all $m \in \mathbb{N}$, and both eigenvalues of its Poincaré return map are positive, so h^m is good for all $m \in \mathbb{N}$.

Finally, prior to a further perturbation of $K|_{(U' \setminus U)_\phi}$ we have S^1-families of Type $\mathfrak{T}$-orbits indexed by (n, m), and on $(U' \setminus U)_\phi$

$$DR = \frac{K}{(K - qK_q)^2} \begin{bmatrix} 0 & 0 \\ -K_{qq} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\epsilon & 0 \end{bmatrix}, \quad 0 < \epsilon \ll 1. \tag{23}$$

A slight perturbation of K will deform each S^1-family of $\gamma_{n/m}$ into a pair of Reeb orbits: $e_{n/m}$ and $h_{n/m}$. One can show that the Poincar'e return maps of $e_{n/m}$'s are negative rotations, hence has no real eigenvalues, and the Poincar'e return maps for $h_{n/m}$'s are hyperbolic with two positive eigenvalues. So $e_{n/m}$ and $h_{n/m}$ are good as well. ∎

Chern class of ξ and contractible orbits. Identify Σ with the page zero $\Sigma \times \{0\}$ of the mapping torus Σ_ϕ. Let $z \in \Sigma \setminus U$ be a fixed point of ϕ with $\{z\} \times S^1_t$ contractible in M. Let $t_z := [\{z\} \times S^1_t] \in \pi_1(\Sigma_\phi, z)$. Let $\phi_\#$ be the map on $\pi_1(\Sigma, x)$ induced by ϕ. We have the following proposition concerning the topology of Σ_ϕ and M.

Proposition 4.3.

- $\pi_1(\Sigma_\phi) = \langle x \in \pi_1(\Sigma, z), t_z \mid \phi_\# x = t_z^{-1} x t_z \rangle$
- $\pi_1(M) = \langle x \in \pi_1(\Sigma, z) \mid \phi_\# x = x \rangle = \langle a, b \mid b^\sigma = 1 \rangle$
- $H_1(M, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_\sigma$.

Note that $[\Gamma] = b$ and $[\{z\} \times S^1_t] = t_z$ generates an abelian subgroup $\mathbb{Z} \times \mathbb{Z}$ of $\pi_1(\Sigma_\phi)$, as well as an $\mathbb{Z}_\sigma \times \mathbb{Z}$ subgroup of $H_1(\Sigma_\phi)$.

Lemma 4.2. *Let $\xi := \ker \alpha$. Then $c_1(\xi) = 0$ on $H_2(M, \mathbb{Z})$. Hence the μ-index and $\bar{\mu}$-index for homologously trivial Reeb orbits are well defined.*

Proof. Apply Van Kampen theorem to $M = \Sigma_\phi \cup S^1 \times D^2$ we have $H_2(M, \mathbb{R}) \cong H_2(\Sigma_\phi, \mathbb{R})$. This can be seen from the following long exact sequence:

$$\cdots \to H_2(T^2) \xrightarrow{i_2} H_2(\Sigma_\phi) \oplus H_2(S^1 \times D^2) \xrightarrow{j_2} H_2(M) \xrightarrow{\delta}$$
$$H_1(T^2) \xrightarrow{i_1} H_1(\Sigma_\phi) \oplus H_1(S^1 \times D^2) \xrightarrow{j_1} H_1(M) \to \cdots$$

Since $\mathrm{im}(i_2) = 0$ and $j_2 = 0$ on $H_2(S^1 \times D^2)$, the inclusion $H_2(\Sigma_\phi) \hookrightarrow H_2(M)$ is injective. Now, $H_1(T^2)$ is generated by $[t]$ and $[B]$, and $i_1[t] = 0$ in $H_1(S^1 \times D^2)$, $i_1[t] \in H_1(\Sigma_\phi)$ is an element of infinite order; while $i_1[B] = 0$ in $H_1(\Sigma_\phi)$, $i_1[B] \in H_1(S^1 \times D^2)$ is a generator. So i_1 is injective, then $\mathrm{im}(\delta) = 0$, $\mathrm{im}(j_2) = \ker(\delta) = H_2(M)$, and hence the inclusion $H_2(\Sigma_\phi) \hookrightarrow H_2(M)$ is surjective as well. Hence $H_2(M, \mathbb{R}) \cong H_2(\Sigma_\phi, \mathbb{R})$ is generated by $\Gamma_\phi = \Gamma \times S^1_t$. Note that $\xi|_{\Gamma_\phi}$ is S^1_t-invariant, $\xi \cap T(\Gamma_\phi)$ generates a nonsingular 1-dimensional foliation, hence $c_1(\xi)[\Gamma_\phi] = 0$ and we conclude that $c_1(\xi) = 0$ on $H_2(M, \mathbb{Z})$. ☐

Lemma 4.3. 1. *Contractible Reeb orbits are*

- $e_{k\sigma/m}$ *and* $h_{k\sigma/m}$ *with* $k, m \in \mathbb{N}$, $0 < k < m$;
- h^m, $m \in \mathbb{N}$; *and*
- B^m, $m \in \mathbb{N}$, *provided that* $\sigma = 1$ *(note: B^m is only homologously trivial but not contractible if $\sigma > 1$).*

2. *If n is not divisible by σ then $[e_{n/m}] = [h_{n/m}] = (0, j) \in H_1(M, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_\sigma$ where $0 \neq j \in \mathbb{Z}_\sigma$, $j \equiv n \mod \sigma$.*

Moduli of planes and $\partial^2 = 0$. Now the set of contractible Reeb orbits is identified. To determine whether or not the cylindrical contact homology is defined, i.e., whether or not $\partial^2 = 0$, we need to find all of the nonempty moduli $\mathcal{M}(\gamma)$ with γ contractible and $\dim \mathcal{M}(\gamma) = 1$.

Recall that if γ is homotopically trivial and if $c_1(\xi) = 0$ then

$$\dim \mathcal{M}(\gamma) = \bar{\mu}(\gamma). \tag{24}$$

So $\dim \mathcal{M}(\gamma) = 1$ implies that γ is hyperbolic. Since B^m is elliptic for all $m \in \mathbb{N}$ we have $\bar{\mu}(B^m) \equiv \bar{\mu}(B^m, \mathbb{Z}_2) = 0 \mod 2$ and hence the following

Lemma 4.4. $\dim \mathcal{M}(B^m) = $ *even if* $\mathcal{M}(B^m) \neq \emptyset$.

Remark 4.3. *In fact, $\bar{\mu}(B^m)$ can be made arbitrarily large by taking the constant c in Lemma 3.1 to be large enough.*

Lemma 4.5. $\mathcal{M}(e_{k\sigma/m}) = \emptyset = \mathcal{M}(h_{k\sigma/m})$ *for* $k, m \in \mathbb{N}$, $0 < k < m$.

Proof. Let $\gamma = e_{k\sigma/m}$ or $h_{k\sigma/m}$ with $k, m \in \mathbb{N}$, $0 < k < m$. Assume that $\mathcal{M}(\gamma) \neq \emptyset$ and let C_M be the image in M of an element of $\mathcal{M}(\gamma)$. Since $m = \mathrm{wind}(\gamma, B) \geq 1$ so $C_M \cap B \neq \emptyset$, C_M intersects positively with B at every point of intersection. We write $C \cap B = \{z_i\}_{i=1}^s$ where $z_1, ..., z_s$ are s distinct points. Let m_i denote the intersection multiplicity at z_i, then $m_i > 0$ and $\sum_{i=1}^s m_i = m$. Let U_B be a thin tubular neighborhood of B. We may assume that $C \pitchfork \partial U_B$ and $C \cap U_B$ consists of s disjoint discs $D_i \ni z_i$. Let $C' := C_M \setminus U_B$. We have $\partial C' = \gamma \cup (\cup_{i=1}^s (-\partial D_i))$. Note that each for reach i, $[\partial D_i] = [t]^{m_i} \in \pi_1(\Sigma_\phi)$, while $[\gamma] = [p]^{k\sigma}[t]^m \in \pi_1(\Sigma_\phi)$. By Propositions 4.1 and 4.3 the existence of C_M and hence C' implies that $[p]^{k\sigma} = 0 \in \pi_1(\Sigma_\phi)$ which is impossible unless $m\sigma$ divides $k\sigma$. By assumption $0 < k < m$, $m\sigma$ does not divide $k\sigma$. So C' does not exist. $\mathcal{M}(\gamma)$ is empty. ∎

Now consider h the simple Type $\mathfrak{C}$ Reeb orbit. We can find a spanning disc D_h for h so that $D_h \cap B$ is a point, ξ_{D_h} has only one singularity and it is elliptic. A simple calculation will yield the following

Lemma 4.6. *For* $m \in \mathbb{N}$, $\dim \mathcal{M}(h^m) = \bar{\mu}(h^m) = 2m - 1$ *provided that* $\mathcal{M}(h^m) \neq \emptyset$.

So $\dim \mathcal{M}(h^m) = 1 \Rightarrow m = 1$. One can show that, as the S^1-invariant case considered in [24], $\mathcal{M}(h)/\mathbb{R}$ is in one-one correspondence with the gradient trajectories from the corresponding hyperbolic critical point x_h of K to $\partial\Sigma$. There are two such trajectories and they are counted with opposite signs.

Hence we have

Lemma 4.7. $d_0 h = 0$. *(See Remark 2.1 for the definition of d_0.)*

Remark 4.4. Recall that to show $\partial^2 = 0$ it suffices to prove that $d_0 = 0$. Our computation above implies that indeed $d_0 = 0$ and hence $\partial^2 = 0$. The cylindrical contact homology is then defined. This will be confirmed again in Section 4.5 (see Lemma 4.15 and Corollary 4.2).

4.3 Holomorphic cylinders in $\mathbb{R} \times M$

To compute the cylindrical contact homology of (M, ξ) we need to find in $\mathbb{R} \times M$ all 1-dimensional moduli $\mathcal{M}(\gamma_-, \gamma_+)$ of holomorphic cylinders converging to Reeb orbit γ_- at $-\infty$ and to Reeb orbit γ_+ at ∞. First we find all ordered pairs (γ_-, γ_+) such that $\dim \mathcal{M}(\gamma_-, \gamma_+) = 1$ if $\mathcal{M}(\gamma_-, \gamma_+) \neq \emptyset$, then we count elements of $\mathcal{M}(\gamma_-, \gamma_+)/\mathbb{R}$.

Type $\mathfrak{B}$ orbits. The homotopy classes of Reeb orbits imply the following

Lemma 4.8. *$\mathcal{M}(\gamma, B^m) = \emptyset = \mathcal{M}(B^m, \gamma)$ for all $m \in \mathbb{N}$ if γ is a Type $\mathfrak{T}$ or $\mathfrak{C}$ orbit.*

Proof. Let $\mathcal{M} = \mathcal{M}(\gamma, B^m)$ or $\mathcal{M}(B^m, \gamma)$. Assume that $\mathcal{M} \neq \emptyset$ and let C_M be the image in M of an element of $\mathcal{M}$. C_M is a homotopy between γ and B^m. Since γ and B^m are not homotopic in Σ_ϕ, C_M has to intersect B. We may assume that the intersection is transversal. By holomorphicity C_M intersects positively with B at every point of $C_M \cap B$. The total multiplicity of the intersection is $m = \mathrm{wind}(\gamma, B)$. Thus we have $[\gamma][t]^{-m} = [B]$ in $\pi_1(\Sigma_\phi)$. But $[\gamma][t]^{-m}$ is in the abelian subgroup of $\pi_1(\Sigma_\phi)$ generated by $b = [\Gamma]$ while $[B] = aba^{-1}b^{-1}$ is not in this abelian subgroup, a contradiction to the existence of C_M. So $\mathcal{M} = \emptyset$. $\square$

Lemma 4.9. $\dim \mathcal{M}(B^n, B^m) = \bar{\mu}(B^m) - \bar{\mu}(B^n)$ *is an even number provided that $\mathcal{M}(B^n, B^m) \neq \emptyset$.*

Recall that $\bar{\mu}(B^m)$ is always an even number and can be made arbitrarily large by modifying α. This property together with Lemma 4.4 and Lemma 4.9 shows that B^m has eventually no contribution to the boundary operator ∂ of the cylindrical contact homology and hence can be neglected. We will then focus on Type $\mathfrak{T}$ and $\mathfrak{C}$ Reeb orbits and assume that $\gamma_\pm \neq B^m$, $\forall m \in \mathbb{N}$.

Going upstairs. Let

$$C \in \mathcal{M}(\gamma_-, \gamma_+)$$

denote any one of such cylinders. After a 0-surgery along B, C will become a punctured holomorphic sphere $\hat{C} \subset \mathbb{R} \times \hat{M}$ with one positive puncture and $1 + s$ negative punctures, $s \geq 0$.

Recall that a 0-surgery along B turns the open book $M = (\Sigma, \tau)$ into the mapping torus $\hat{M}$ with B replaced by the orbit $\hat{e}$ corresponding to the extra elliptic fixed point of $\hat{\phi}'$ on $D^2 = T^2 \setminus \Sigma$; $e_{n/m}$, $h_{n,m}$ and h^m $(0 < n < \sigma m,$

$n, m \in \mathbb{N}$) are lifted to $\hat{M}$ and are now denoted by $\hat{e}_{n/m}$, $\hat{h}_{n/m}$ and $\hat{h}^m$ to distinguish them from their copies in M.

Surely the new holomorphic curve $\hat{C}$ converges to $\hat{\gamma}_+$ at ∞ and its original negative puncture converges to $\hat{\gamma}_-$ at $-\infty$. The number s of extra negative punctures is equal to the the number of geometric intersection points (not counting multiplicity) of C with $\mathbb{R} \times B$. Each of these punctures converges to $\hat{e}^{m_i}$ at $-\infty$ for some $m_i \in \mathbb{N}$. In other words,

$$\hat{C} \in \mathcal{M}(\{\hat{\gamma}_-, \hat{e}^{m_1}, ..., \hat{e}^{m_s}\}, \hat{\gamma}_+).$$

m_i is the intersection multiplicity of C and $\mathbb{R} \times B$ at the corresponding point.

Definition 4.1 (winding number around B). Suppose that $\gamma = h_{n/m}$ or $e_{n/m}$ or h^m. We call the number m the *winding number* of γ around B, and denote it by $\mathrm{wind}(\gamma, B)$.

We have

$$\sum_{i=1}^{s} m_i = \mathrm{wind}(\gamma_+, B) - \mathrm{wind}(\gamma_-, B). \tag{25}$$

Note that γ_+ and γ_- are free homotopic in M.

One dimensional moduli of cylinders. By using the homotopic property of Reeb orbits we can obtain the following

Proposition 4.4. *Let $\gamma_\pm$ be distinct Reeb orbits of Type $\mathfrak{T}$ or Type $\mathfrak{C}$. Suppose that $\mathcal{M}(\gamma_-, \gamma_+) \neq \emptyset$ and $\dim \mathcal{M}(\gamma_-, \gamma_+) = 1$ then (γ_-, γ_+) must be one of the following ordered pairs:*

1. $(e_{n/m}, h_{n/m})$, $0 < n < \sigma m$;
2. $(h_{n/(m-1)}, e_{n/m})$, $0 < n < \sigma(m-1)$;
3. $(h^{m-1}, e_{(\sigma m-1)/m})$;
4. $(h_{(n-\sigma)/(m-1)}, e_{n,m})$, $\sigma < n < \sigma m$;
5. $(h^{m-1}, e_{\sigma/m})$.

Corollary 4.1. $\mathcal{C}^o_*(\alpha) = 0$ *for* $* \leq 0$.

Proof of Corollary 4.1. Recall that $\mathcal{C}^o(\alpha)$ is generated by (1) B^m with $m \in \mathbb{N}$ if $\sigma = 1$ and (2) h^m with $m \in \mathbb{N}$, (3) $e_{k\sigma/m}$ and $h_{n/m}$ with $k, m \in \mathbb{N}$ and

$0 < k < m$. We have known that $\bar{\mu}(B^m) \gg 1$ and $\bar{\mu}(h^m) = 2m-1 \geq 1$ for $m \in \mathbb{N}$. Now, Part 1 of Proposition 4.4 implies that

$$\bar{\mu}(h_{k\sigma/m}) = 1 + \bar{\mu}(e_{k\sigma/m}).$$

By applying Part 2-5 of Proposition 4.4 several times we get

$$\bar{\mu}(e_{k\sigma/m}) = 1 + \bar{\mu}(h^{m-1}) = 2m-2 \geq 2 \quad \text{for } k, m \in \mathbb{N}, 0 < k < m, \tag{26}$$

and hence $\bar{\mu}(h_{k\sigma/m}) = 2m-1 \geq 3$ for $k, m \in \mathbb{N}$ and $0 < k < m$.

This proves that $\mathcal{C}^o_*(\alpha) = 0$ for $* \leq 0$. □

The rest of this section up to Lemma 4.12 is devoted to the proof of Proposition 4.4.

Lemma 4.10. *Assume that $\gamma_\pm$ are distinct Reeb orbits of Type $\mathfrak{T}$ or Type $\mathfrak{C}$. Suppose that γ_- and γ_+ are free homotopic in Σ_ϕ and the formal dimension of $\mathcal{M}(\gamma_-, \gamma_+)$ equals 1, then $(\gamma_-, \gamma_+) = (e_{n/m}, h_{n/m})$.*

Suppose that γ_- and γ_- are not free homotopic in Σ_ϕ and $\mathcal{M}(\gamma_-, \gamma_+) \neq \emptyset$. Let $C \in \mathcal{M}(\gamma_-, \gamma_+)$ be a holomorphic cylinder. Then $C \cap (\mathbb{R} \times B) \neq \emptyset$. Since C and $\mathbb{R} \times B$ are holomorphic of complement dimensions, they intersects positively at every point of intersection. Let C_M denote the image of C in M.

Claim: $C_M \cap h = \emptyset$.

Proof of Claim: Let $L := \gamma_+ \cup (-\gamma_-)$. $[L] = [h] = 0 \in H_1(M, \mathbb{Z})$ hence the linking number $lk(L, h) = lk(h, L) \in \mathbb{Z}$ is defined. Let S be any surface with boundary $\partial S = L$. Then $lk(L, h) = S \cdot h$ is the algebraic number of intersection points of S with h. $lk(L, h)$ is independent of the choice of the spanning surface S of L (sine h is homologously trivial). In particular, $C_M \cdot h = lk(L, h)$. On the other hand, we can find an embedded spanning disc D of h such that $D \cap B$ is a single point and $D \cdot L = \emptyset$. So $C_M \cdot h = lk(h, L) = D \cdot L = 0$. Hence $C \cdot (\mathbb{R} \times h) = C_M \cdot h = 0$. C and $\mathbb{R} \times h$ are holomorphic of complement dimensions hence intersects positively at each point of intersection, thus we must have $C \cap (\mathbb{R} \times h) = \emptyset = C_M \cap h$. □

Lemma 4.11. *Assume that $\gamma_\pm$ are Reeb orbits of Type $\mathfrak{T}$ or Type $\mathfrak{C}$. Then*

$$\dim \mathcal{M}(\gamma_-, \gamma_+) = r + 2(wind(\gamma_+, B) - wind(\gamma_-, B)),$$

where

1. $r = 0$ *if* $\gamma_\pm$ *are both elliptic or hyperbolic;*

2. $r = 1$ *if* γ_+ *is hyperbolic and* γ_- *is elliptic;*

3. $r = -1$ *if* γ_+ *is elliptic and* γ_- *is hyperbolic.*

Proof. Recall that $C_M \cap B$ is a finite set of points say $z_1, ..., z_s$. Let m_i denote the multiplicity of z_i, then $\sum_{i=1}^s m_i = \text{wind}(\gamma_+, B) - \text{wind}(\gamma_-, B) \geq 0$. Let U_B denote a tubular neighborhood of $B \subset M$ so that $\partial U_B \pitchfork C_M$ and $U_B \cap C_M$ is a union of s disjoint discs.

On $\hat{\Sigma} = T^2$ take a nonvanishing vector field Z_o and perturb it slightly to get a new vector field Z such that Z has an elliptic singularity at the critical point $x_e \in Crit(\hat{K})$ corresponding to $\hat{e}$, a hyperbolic singularity at the saddle point $x_h \in Crit(\hat{K})$ corresponding to $\hat{h}$, and Z is nonvanishing elsewhere. Since the volume form of $\Sigma = \hat{\Sigma} \setminus D^2$ is positive on ξ, Z defines a symplectic trivialization Ψ on $M \setminus (U_B \cup h)$. The $\bar{\mu}$-index of $\gamma = \gamma_\pm$ with respect to Ψ is

$$\bar{\mu}_\Psi(\gamma) = \begin{cases} -1 & \text{if } \gamma = h_{n/m} \text{ or } h^m, \\ -2 & \text{if } \gamma = e_{n/m}. \end{cases} \tag{27}$$

Let Ψ' be a symplectic trivialization of $U_B \cong B \times D^2$ which is invariant under rotations in B. Let $D_i \subset (C \cap U_B)$ denote the connected component containing z_i, $i = 1, ..., s$. Consider along ∂D_i the 1-dimensional subbundle $\mathcal{L}_i := \xi \cap T_{D_i}C$. $\mathcal{L}_i$ is homotopic to the 1-dimensional subbundle spanned by the radio vector field $\partial_r|_{\partial D_i}$. Under the trivialization Ψ the bundle $\mathcal{L}_i$ becomes a loop in the Lagrangian Grassmann $\Lambda(\mathbb{R}^2)$ with Maslov index $n_i = 0$. While with respect to Ψ' the corresponding loop has Maslov index $n'_i = 2m_i$ which is twice of the multiplicity of z_i. Thus we have (provided that $\mathcal{M}(\gamma_-, \gamma_+) \neq \emptyset$)

$$\begin{aligned} \dim \mathcal{M}(\gamma_-, \gamma_+) &= \bar{\mu}_\Psi(\gamma_+) - \bar{\mu}_\Psi(\gamma_-) + \sum_{i=1}^s (n'_i - n_i) \\ &= \bar{\mu}_\Psi(\gamma_+) - \bar{\mu}_\Psi(\gamma_-) + \sum_{i=1}^s 2m_i \\ &= \bar{\mu}_\Psi(\gamma_+) - \bar{\mu}_\Psi(\gamma_-) + 2(\text{wind}(\gamma_+, B) - \text{wind}(\gamma_-, B)). \end{aligned} \tag{28}$$

Note that $|\bar{\mu}_\Psi(\gamma_+) - \bar{\mu}_\Psi(\gamma_-)| \leq 1$ and $\text{wind}(\gamma_+) - \text{wind}(\gamma_-) \geq 0$. So if $\dim \mathcal{M}(\gamma_-, \gamma_+) = 1$ then one of the followings must be held true:

1. γ_+ is hyperbolic, γ_- is elliptic and $\mathrm{wind}(\gamma_+, B) = \mathrm{wind}(\gamma_-, B)$.

2. γ_+ is elliptic, γ_- is hyperbolic and $\mathrm{wind}(\gamma_+, B) = 1 + \mathrm{wind}(\gamma_-, B)$.

This settles the denominators (i.e, winding numbers around B) of $\gamma_\pm$.

It is easy to see that $\gamma_+ \neq h^m$, $\forall m \in \mathbb{N}$. The difference of the numerators of $\gamma_\pm$ is divisible by σ since γ_+ and γ_- are free homotopic. Assume that $\gamma_+ = e_{n/m}$ and $\gamma_- = h_{n'/(m-1)}$ with $n - n' = k\sigma$. Note that notationally $h_{(im\sigma)/m} = h^m$ for all i. Then k is the algebraic number of the times that the image $C_M \subset M$ of $C \in \mathcal{M}(\gamma_-, \gamma_+)$ crosses Γ_ϕ. Since C_M intersects B only once, we have $k = 0$ or 1. This completes the proof. □

Counting cylinders (up to signs). Let

$$\kappa_{n,m} := \gcd(n, m)$$

denote the greatest common divisor of n and m. Recall that κ_γ denotes the multiplicity of the Reeb orbit γ, so

$$\kappa_\gamma = \kappa_{n,m} \quad \text{if } \gamma = e_{n/m} \text{ or } h_{n/m}.$$

Lemma 4.12. *Let $(\gamma_-, \gamma_+) = (h_{n'/(m-1)}, e_{n/m})$, $0 < n < \sigma m$, $0 \leq n'$, $n' = n$ or $n - \sigma$, be equal to one of four types of the ordered pairs in Proposition 4.4. Here $h_{0/(m-1)}$ represents the orbit h^{m-1}. Then the moduli $\mathcal{M}(\gamma_-, \gamma_+)/\mathbb{R}$ consists of*

$$\frac{|n'm - n(m-1)|}{\kappa_{n',m-1}\kappa_{n,m}}$$

points. All the corresponding pseudoholomorphic cylinders are immersed *surfaces in $\mathbb{R} \times M$.*

Remark 4.5. It will be proved later that a coherent orientation can be defined so that elements of the same moduli $\mathcal{M}(h_{n'/(m-1)}, e_{n/m})$ have the same orientation.

The rest of this section will be occupied by the proof of Lemma 4.12. We will need Taubes's [21] description on trice-punctured spheres, Bourgeois's [1] Morse-Bott version of curve counting and some arguments brought from Hutchings-Sullivan's paper [15].

More surgery models of $\hat{M} = \hat{\Sigma}_{\hat{\phi}}$. Below we describe a more symmetric construction of $\hat{M}$ to compare with the contact structure on $S^2 \times S^1$ considered in [21].

Recall that $\hat{\Sigma} = T^2$ and $\hat{\phi} \in \mathrm{Symp}(T^2, \hat{\omega})$. We can choose $\hat{\omega}$ so that

$$\hat{\omega} = dq \wedge dp$$

with respect to some suitable coordinates $(q, p) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ of T^2. Then $\hat{\phi}$ is isotopic to $\psi \in \mathrm{Symp}(T^2, \hat{\omega})$, where

$$\psi : T^2 \to T^2, \quad \psi(q, p) := (q, p - \sigma q).$$

The 2-form $\hat{\omega} \in \Omega^2(T^2)$ canonically extends to a 2-form on T^2_ψ which is also denoted by $\hat{\omega}$. Note that $d\alpha \in \Omega^2(\Sigma_\phi)$ extends to $\hat{\omega} + \eta \wedge dt$ on T^2_ψ for some closed 1-form $\eta \in \Omega^1(T^2)$.

Here is another way to think of $\hat{M} = T^2_\psi$. Think of S^1_q as the interval $[-1, 0]$ with two endpoints identified. Recall that the $q = 0$ level is the mapping torus $(\{0\} \times S^1_p)_\psi$. Let $V = (-1, 0)_q \times S^1_p$. Let $Z := [-1, 0] \times S^1 \times S^1$. Following [15] we have the identification:

$$\begin{aligned} \Psi : V_\psi &\to (-1, 0) \times S^1 \times S^1 = \mathring{Z}, \\ [(q, p), t] &\to (q, p - \sigma q t, t). \end{aligned}$$

Then

$$T^2_\psi \overset{\Psi}{\approx} \frac{[-1, 0] \times T^2}{(-1, g(p, t) := (p + \sigma t, t)) \sim (0, (p, t))}, \tag{29}$$

and the equality

$$\hat{\omega} = \Psi^* \omega_Z, \quad \omega_Z := dq \wedge (dp + \sigma q dt),$$

extends over T^2_ψ.

Orbits on Z and in T^2_ψ. The vector field $X_Z := \partial_t - \sigma q \partial_p$ generates the line field $\ker(\omega_Z)$. Orbits of X_Z comes in S^1-families parametrized by

$$\mathcal{S}_Z := \{(n, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{N} \mid n \leq \sigma m \}. \tag{30}$$

Notation 4.1. Let $\hat{\Upsilon}_{n/m}$ denote the S^1-family of orbits in Z with index (n, m).

The following facts are obvious:

1. The total space of $\hat{\Upsilon}_{n/m}$ is a $\kappa_{n,m}$-cover of the torus $\{q = \frac{n}{\sigma m}\}$.

2. For $\gamma \in \hat{\Upsilon}_{n/m}$, $[\gamma] = (n, m) \in H_1(S^1_p \times S^1_t, \mathbb{Z})$.

3. With the gluing map g in (29) the two sets $\hat{\Upsilon}_{\sigma m/m}$ and $\hat{\Upsilon}_{0/m}$ are identified in T^2_ψ, and S^1-families of orbits on T^2_ψ are indexed by the set

$$\hat{\mathcal{S}} := \{(n, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{N} \mid 0 < n < \sigma m\}. \tag{31}$$

Compare $\hat{\mathcal{S}}$ with $\mathcal{S}$ in (22) and one finds that $\hat{\mathcal{S}}$ and $\mathcal{S}$ are equal except the extra elements $(0, m) \in \hat{\mathcal{S}}$ whose corresponding S^1-families of orbits will deform to the pair of orbits $\hat{h}^m, \hat{e}^m$ under a perturbation.

Trice-punctured spheres in $\mathbb{R} \times Z$ and in $\mathbb{R} \times T^2_\psi$. Recall that in [21] (Thm. A.2) Taubes classified moduli of trice-punctured pseudoholomorphic spheres in the symplectization $\mathbb{R} \times (S^2 \times S^1)$ equipped with an almost complex structure associated to a certain contact structure on $S^2 \times S^1$.

Our manifold $Z = ([-1, 0] \times S^1) \times S^1$ can be suitably identified with a subset of $S^2 \times S^1$. Note that with a constant $Q > 0$ large enough, the 1-form $\alpha_Z := q dp + (Q + \frac{1}{2}\sigma q^2) dt \in \Omega^1(Z)$ is contact and $d\alpha_Z = \omega_Z$, with Reeb vector field $R_Z = (Q - \frac{1}{2}\sigma q^2)^{-1} X_Z$. Then with a suitable embedding of Z into $S^2 \times S^1$, Taubes's contact 1-form on $S^2 \times S^1$ restricts to $\alpha_T := e^{\rho(q)} \alpha_Z$ on Z for some $\rho \in C^\infty(Z)$ depending only on q. Also, up to a perturbation, α_T and α_Z have the same set of S^1-families of Reeb orbits indexed by (30), the ordering of the orbits are preserved, and both orbit sets are invariant under rotations generated by the group $S^1_p \times S^1_t$.

An energy inequality by Hutchings-Sullivan (see Sec. 3.3 of [15]) confirms that trice-punctured spheres converging to orbits in Z stay in $\mathbb{R} \times Z$. Thus Taubes's result applies to (Z, α_Z).

Consider the triples $(n, m), (n', m'), (n'', m'') \in \mathcal{S}_Z$ with $(n, m), (n', m')$ linearly independent and

$$n = n' + n'', \quad m = m' + m''. \tag{32}$$

Also use the notations

$$\hat{\Upsilon}_+ := \hat{\Upsilon}_{n/m}, \quad \hat{\Upsilon}_- := \hat{\Upsilon}_{n'/m'}, \quad \hat{\Upsilon}_* := \hat{\Upsilon}_{n''/m''}.$$

Recall that

$$\mathcal{M}(Z; \hat{\Upsilon}_*, \hat{\Upsilon}_-; \hat{\Upsilon}_+)$$

denote the moduli space of trice-punctured spheres in $\mathbb{R}\times Z$ with one positive end converging to an element of $\hat{\Upsilon}_+$, one negative end converging to an element of $\hat{\Upsilon}_-$, and one negative end converging to an element of $\hat{\Upsilon}_*$.

Proposition 4.5 ([15] Sec. 3.3). *The total space of* $\mathcal{M}(Z;\hat{\Upsilon}_*,\hat{\Upsilon}_-;\hat{\Upsilon}_+)/\mathbb{R}$ *is contained in* Z *and is between* $\{q=\frac{n'}{\sigma m'}\}$ *and* $\{q=\frac{n''}{\sigma m''}\}$.

Remark 4.6. Proposition 4.5 ensures that $\mathcal{M}(Z;\hat{\Upsilon}_*,\hat{\Upsilon}_-;\hat{\Upsilon}_+)$ is preserved when boundaries of Z are glued by g to get T^2_ψ.

Proposition 4.6 ([21] Thm. A.2). $\mathcal{M}(Z;\hat{\Upsilon}_*,\hat{\Upsilon}_-;\hat{\Upsilon}_+)/\mathbb{R}$ *is connected and diffeomorphic to* $S^1\times S^1$. *The rotation group* $G:=S^1_p\times S^1_t$ *induces a free action on* $\mathcal{M}(Z;\hat{\Upsilon}_*,\hat{\Upsilon}_-;\hat{\Upsilon}_+)/\mathbb{R}$ *with the orbit space a single point. Every element of* $\mathcal{M}(Z;\hat{\Upsilon}_*,\hat{\Upsilon}_-;\hat{\Upsilon}_+)$ *is an immersed curve.*

Counting in the Morse-Bott way. Upon a perturbation using a suitable Morse function f (see [1]), each S^1-family of orbits deforms to a pair of orbits: one elliptic and one hyperbolic.

Let

$$\hat{e}_*,\ \hat{e}_-,\ \hat{e}_+,\quad \hat{h}_*,\ \hat{h}_-,\ \hat{h}_+$$

denote the elliptic and hyperbolic elements of $\hat{\Upsilon}_*,\hat{\Upsilon}_-,\hat{\Upsilon}_+$ respectively.

We are interested in counting elements of 0-dimensional moduli of the type $\mathcal{M}(Z;\hat{e}_*,\hat{h}_-;\hat{e}_+)/\mathbb{R}$.

Proposition 4.7 ([1] Sec. 3.2, 3.3). *Assume that* $\mathcal{M}(Z;\hat{e}_*,\hat{h}_-;\hat{e}_+)/\mathbb{R}\neq\emptyset$ *and* $\dim\mathcal{M}(Z;\hat{e}_*,\hat{h}_-;\hat{e}_+)/\mathbb{R}=0$. *Then* $\mathcal{M}(Z;\hat{e}_*,\hat{h}_-;\hat{e}_+)/\mathbb{R}$ *is equal to the fibered product*

$$\Big(W^s(\hat{e}_*)\times W^s(\hat{h}_-)\times_{(\hat{\Upsilon}_*\times\hat{\Upsilon}_-)}\mathcal{M}(Z;\hat{\Upsilon}_*,\hat{\Upsilon}_-;\hat{\Upsilon}_+)\times_{\hat{\Upsilon}_+}W^u(\hat{e}_+)\Big)/\mathbb{R},$$

where $W^s(\cdot)$, $W^s(\cdot)$ *are respectively the corresponding stable and unstable submanifolds of the Morse function; and the maps involved in the fibered product are evaluation maps.*

Note that $W^u(\hat{e}_+)=\{\hat{e}_+\}$, $W^s(\hat{h}_-)=\{\hat{h}_-\}$ and $W^s(\hat{e}_*)=\hat{\Upsilon}_*\setminus\{\hat{h}_*\}$.

Remark 4.7. As the perturbation caused by the Morse function f is small, elements of $\mathcal{M}(Z;\hat{e}_*,\hat{h}_-;\hat{e}_+)$ are immersed curves, following Proposition 4.6.

Remark 4.8. it will be shown in Section 4.5 that elements of the same moduli $\mathcal{M}(Z;\hat{e}_*,\hat{h}_-;\hat{e}_+)$ have the same sign, hence up to sign the algebraic number of $\mathcal{M}(Z;\hat{e}_*,\hat{h}_-;\hat{e}_+)/\mathbb{R}$ equals the geometric number of $\mathcal{M}(Z;\hat{e}_*,\hat{h}_-;\hat{e}_+)/\mathbb{R}$.

To us the relevant cases are

$$* = \begin{cases} (0, m-1) & \text{if } 0 < n \le \sigma(m-1), \\ (\sigma, m-1) & \text{if } \sigma \le n < \sigma m. \end{cases} \tag{33}$$

Then the corresponding 1-dimensional spaces are

1. $\mathcal{M}(Z; \hat{e}_{0/(m-1)}, \hat{h}_{n/(m-1)}; \hat{e}_{n/m})$ for $0 < n \le \sigma(m-1)$;

2. $\mathcal{M}(Z; \hat{e}_{\sigma/(m-1)}, \hat{h}_{(n-\sigma)/(m-1)}; \hat{e}_{n/m})$ for $\sigma \le n < \sigma m$.

Recall the group G from Proposition 4.6. Apply the group G to the evaluation map

$$ev : \mathcal{M}(Z; \hat{\Upsilon}_*, \hat{\Upsilon}_-; \hat{\Upsilon}_+)/\mathbb{R} \to \hat{\Upsilon}_{*-+} := \hat{\Upsilon}_* \times \hat{\Upsilon}_- \times \hat{\Upsilon}_+.$$

We find that the image class $[ev(\mathcal{M}/\mathbb{R})] \in H_2(\hat{\Upsilon}_{*-+}, \mathbb{Z})$ is equal to

$$\pm a[\hat{\Upsilon}_+ \times \hat{\Upsilon}_-] \pm b[\hat{\Upsilon}_+ \times \hat{\Upsilon}_*] \pm c[\hat{\Upsilon}_- \times \hat{\Upsilon}_*],$$

for some $a, b, c \in \mathbb{N}$.

Fact 4.1. *For a general triple $(n, m) = (n', m') + (n'', m'')$ (with $(n, m), (n', m')$ linearly independent) we have $a = |nm' - mn'|(\kappa_{n,m}\kappa_{n',m'})^{-1}$.*

In particular

$$a = \begin{cases} n/(\kappa_{n,m}\kappa_{n,m-1}) & \text{if } * = (0, m-1), \\ (\sigma m - n)/(\kappa_{n,m}\kappa_{n-\sigma,m-1}) & \text{if } * = (\sigma, m-1). \end{cases}$$

Let

$$\hat{\Upsilon}_{*-+} \xrightarrow{\pi_*} \hat{\Upsilon}_*, \quad \hat{\Upsilon}_{*-+} \xrightarrow{\pi_{-+}} \hat{\Upsilon}_- \times \hat{\Upsilon}_+$$

denote the obvious projections. A direct calculation yields the following

Fact 4.2. *Assume that (33) holds. Then*

$$\pi_*\Big(ev(\mathcal{M}(Z; \hat{e}_*, \hat{h}_-; \hat{e}_+)/\mathbb{R}) \cap \pi_{-+}^{-1}(\hat{h}_-, \hat{e}_+)\Big)$$

consists of a single element in $W^s(\hat{e}_*)$.

By Lemma 4.5 holomorphic curve in $\mathbb{R} \times Z$ descent to holomorphic curves in $\mathbb{R} \times T^2_\psi$. Note that

$$\hat{h}_{0/m} = \hat{h}_{\sigma m/m} \quad \text{and} \quad \hat{e}_{0/m} = \hat{e}_{\sigma m/m} \quad \text{in } \hat{M} = T^2_\psi.$$

By Gromov compactness as in [14] (Sec. 9.4), if a further deformation is applied then the numbers $\#\mathcal{M}(\hat{e}_*, \hat{h}_-, \hat{e}_+)/\mathbb{R}$ can change only when broken curves appear during the deformation. To us the only relevant cases are

1. broken curve from $\hat{e}_{n/m}$ to $\hat{h}_{n/(m-1)}\hat{h}$ to $\hat{h}_{n/(m-1)}\hat{e}$, where the latter pieces consists of the trivial cylinder over $h_{n/(m-1)}$ and a cylinder between h and e;
2. broken curve from $\hat{e}_{n/m}$ to $\hat{h}_{(n-\sigma)/(m-1)}\hat{h}$ to $\hat{h}_{(n-\sigma)/(m-1)}\hat{e}$, where the latter pieces consists of the trivial cylinder over $\hat{h}_{(n-\sigma)/(m-1)}$ and a cylinder between $\hat{h}$ and $\hat{e}$.

In both cases the broken curves appear in cancelling pairs (due to the fact that $\mathcal{M}(\hat{e},\hat{h})/\mathbb{R}$ consists of two points with opposite signs, see Lemma 4.7), so they do not affect the algebraic numbers of $\mathcal{M}(\hat{M};\hat{e}_*,\hat{h}_-;\hat{e}_+)/\mathbb{R} = \mathcal{M}(M;\hat{h}_-,\hat{e}_+)/\mathbb{R}$. This completes the proof of Lemma 4.12.

4.4 An energy estimate

Now we are back to the contact 3-manifold (M,ξ). In the following we will derive an energy estimate which will help describe the signs of coefficients of the boundary operator ∂ (see Lemma 4.14 in Section 4.5).

Let $\gamma = \gamma_{n/m}$be an unperturbed Type $\mathfrak{T}$ Reeb orbit with $0 < n < \sigma m$. Recall that $\alpha = qdp + K(q)dt$. Then $\gamma \subset T^2_{q_o} = \{q_o\} \times S^1_p \times S^1_t$ with q_o satisfying $-K_q(q_o) = \frac{n}{m}$. Note that α pullbacks to a *closed* 1-form on $T^2_q \cong T^2$ that depends only on q. Thus for any closed curve $\gamma' \subset T^2_q$, the integral $\int_{\gamma'}\alpha$ depends only on the homology class of γ'. Consider the following homotopy of curves including γ:

$$\Xi : [q_+, q'_+] \times S^1 \to [q_+, q'_+] \times S^1_p \times S^1_t,$$
$$\Xi(q,\theta) = (q, p_o + n\theta, m\theta).$$

We have

$$A_\gamma(q) := \int_{\Xi(q,\cdot)} \alpha = nq + mK(q). \tag{34}$$

Differentiating A_γ with respect to q we have

$$\frac{dA_\gamma}{dq} = n + mK_q(q), \quad \frac{d^2 A_\gamma}{dq^2} = K_{qq}(q) > 0.$$

Note that $K_q(q_o) = \frac{-n}{m}$ and K is a strictly decreasing function of q. $\frac{dA_\gamma}{dq} < 0$ when $q < q_o$, $\frac{dA_\gamma}{dq}(q_o) = 0$, and $\frac{dA_\gamma}{dq} > 0$ when $q > q_o$. q_o is the unique minimum point of A_γ. We have the following

Lemma 4.13. *Let $\gamma = \gamma_{n/m}$be an unperturbed Type $\mathfrak{T}$ Reeb orbit with $0 < n < \sigma m$. $\gamma \subset T^2_{q_o}$. Then the action function $A_\gamma(q)$ is a strictly increasing function of $|q - q_o|$.*

4.5 Orientation

Let J be an α-admissible almost complex structure on $W := \mathbb{R} \times M$ and let $\tilde{u} \in \mathcal{M}(\gamma_-, \gamma_+)$ be a pseudoholomorphic map. $\tilde{u} : \mathbb{R} \times S^1 \to W$. Let $(s, \tau) \in \mathbb{R} \times S^1$, $s + i\tau$ be the complex coordinate. $\tilde{u}$ satisfies the d-bar equation

$$\bar{\partial}\tilde{u} := \tilde{u}_s + J\tilde{u}_\tau.$$

Assume from now on that $\tilde{u}$ is an *immersion*. Then there are two ways of splitting $\tilde{u}^*TW$ into a direct sum of two trivial J-complex line bundles:

$$\begin{aligned} \tilde{u}^*TW &= \tilde{u}^*\underline{\mathbb{C}} \oplus \tilde{u}^*\xi \\ &= T \oplus N, \end{aligned}$$

where $\underline{\mathbb{C}} = \mathrm{Span}(\partial_t, R_\alpha)$, $T = \mathrm{Span}(\partial_s, \partial_\tau) = \tilde{u}^*\mathrm{Span}(\tilde{u}_s, \tilde{u}_\tau)$, and N is the pullback of the normal bundle via $\tilde{u}$. We can compactify $\mathbb{R} \times S^1$ by adding to it two circles $S^1_{\pm\infty}$ at infinities to get $[-\infty, \infty] \times S^1$. Also compactify $\mathbb{R} \times M$ to get $[-\infty, \infty] \times M$. J can be extended over $\{\pm\infty\} \times M$ smoothly ,and $\tilde{u}$ can be extended over $S^1_{\pm\infty}$ pseudoholomorphically. The linearization of $\bar{\partial}$ at $\tilde{u}$ is

$$\begin{aligned} D = D_{\tilde{u}} : W^{1,p}(\mathbb{R} \times S^1, \tilde{u}^*TW) &\to L^p(\mathbb{R} \times S^1, \tilde{u}^*TW), \\ D\eta &= \eta_s + J\eta_t + (\nabla_\eta J)\tilde{u}_\tau. \end{aligned}$$

Trivialize $\tilde{u}^*TW = T \oplus N$ by the frame $\{\partial_s, \partial_\tau, \nu, J\nu\}$ with $\nu, J\nu \in N$. Then

$$D\eta = \eta_s + J\eta_t + S(s, \tau)\eta,$$

where $S(s, \tau) \in \mathbb{R}^{4\times 4}$ is of the block form

$$\begin{bmatrix} O & S_1 \\ O & S_2 \end{bmatrix}$$

with each block a 2×2 matrix. Note that

$$S_1 = O \text{ and } S_2 = -J\nabla R_\alpha \quad \text{at } s = \pm\infty.$$

Since S_2 is symmetric at $s = \pm\infty$, modulo some compact perturbation, we may as well assume that $S_1 = O$ and S_2 is symmetric for all s and τ.

Write $D = \frac{d}{ds} - L$ where $L = -J\frac{d}{d\tau} - S$. Think of L as an family of self-adjoint operators from $W^{1,p}(S^1, \mathbb{R}^4)$ to $L^p(S^1, \mathbb{R}^4)$. We can write $L = L_T \oplus L_N$ according to the decomposition $T \oplus N$. Then $L_T(s, \cdot) = -i\frac{d}{d\tau}$ for all s, and $L_N(s, \cdot) = -i\frac{d}{d\tau} - S_2(s, \cdot)$ with S_2 symmetric for all s.

Since $D_{\tilde{u}}$ is surjective, $\text{Ind}(D_{\tilde{u}}) = \dim\ker(D_{\tilde{u}}) = \dim \mathcal{M}(\gamma_-, \gamma_+)$, which is the number of positive eigenvalues of $L(-\infty, \cdot)$, counted with multiplicity, flow to negative eigenvalues of $L(\infty, \cdot)$. Note that L_T is static, i.e., independent of s, so the focus is on L_N. For each s, $L_N(s, \cdot)$ is a perturbation of the operator $-i\frac{d}{d\tau}$ by a 2×2 symmetric matrix $S_2(s, \cdot)$. With a trivialization $\{\nu, J\nu\}$ of N fixed, following [13] one can define the *winding number of an eigenvector field of* $L_N(\pm\infty, \cdot)$ *along* $\gamma_\pm$ (relative to a trivialization of the normal bundle). We have the following properties from [13].

- For each s the multiplicity of an eigenvalue of $L_N(s, \cdot)$ is at most 2.
- For each $k \in \mathbb{Z}$ and for each s there are precisely two (not necessarily distinct) eigenvalues $\lambda_k^- \leq \lambda_k^+$ with winding number k, and the map from s to the closed interval $[\lambda_k^-, \lambda_k^+]$ is continuous for every k,

The second property above means that the ordering of eigenvalues with distinct winding numbers is preserved under perturbation, and eigenvalues with distinct winding numbers never meet. This property greatly restricts the behavior of the spectral flow of $s \to L_N(s, \cdot)$.

A *coherent orientation* of the moduli space is a choice of an orientation of the determinant bundle

$$\det(D_{\tilde{u}}) = (\Lambda^{\max} \ker(D_{\tilde{u}})) \otimes (\Lambda^{\max} \text{coker}(D_{\tilde{u}}))^*$$

for all $\tilde{u} \in \mathcal{M}(\gamma_-, \gamma_+)$ and all $\gamma_\pm$, such that these orientations match well under gluing operation (see [9][8][4]). In the context of contact homology theory and symplectic field theory, coherent orientations always exist and may not be unique.

Recall that 0-dimensional moduli spaces must be of the form $\mathcal{M}(\gamma, \gamma)$ and each of such moduli space consists of a single element whose image is $\mathbb{R} \times \gamma$. For $\tilde{u} \in \mathcal{M}(\gamma, \gamma)$, $\ker(D_{\tilde{u}}) = \{0\} = \text{coker}(D_{\tilde{u}})$, $\det(D_{\tilde{u}}) = 1 \otimes 1^*$ has a natural orientation.

Note that for each Reeb orbit γ, the almost complex structure on ξ induces a complex orientation on $\Gamma(\gamma, \xi_\gamma) \cong C^\infty(S^1, \mathbb{R}^2)$. When α is regular, $L_\gamma = -J(\frac{d}{d\tau} - J\nabla R_\alpha|_\gamma)$ has no eigenvalue equal to 0, $\Gamma(\gamma, \xi_\gamma)$ can be written as a direct sum of two infinite dimensional vector spaces

$$\Gamma(\gamma, \xi_\gamma) = V_-(\gamma) \oplus V_+(\gamma),$$

where $V_-(\gamma)$ (resp. $V_+(\gamma)$) is spanned by all negative (resp. positive) eigenvector fields of L_γ.

Given $\tilde{u} \in \mathcal{M}(\gamma_-, \gamma_+)$, then the restriction of $\ker D_{\tilde{u}}$ to γ is a subspace E of $V_+(\gamma_-)$, such that $V_-(\gamma) \oplus E$ is identified with $V_-(\gamma_+)$ via the spectral flow. Certainly E is sent by the flow to the restriction of $\ker D_{\tilde{u}}$ to γ_+, a subspace of $V_-(\gamma_+)$.

We apply the above discussion to our case here. Recall Proposition 4.4, Lemma 4.12 and Lemma 4.15. We need to better understand the signs of terms in $\partial e_{n/m}$ to determine the boundary operator ∂.

Let γ be h^m or a Type $\mathfrak{T}$ Reeb orbit of either the unperturbed case ($\gamma = \gamma_{n/m}$) or the perturbed case ($\gamma = e_{n/m}$ or $h_{n/m}$). γ is the s^{th} iterate of a simple Reeb orbit denoted by γ'. A tubular neighborhood of γ' is diffeomorphic to $S^1 \times D^2$, with γ' identified with $S^1 \times \{0\}$. Here we identify S^1 with $\mathbb{R}/a'\mathbb{Z}$ where $a' = \mathcal{A}(\gamma') = \int_{\gamma'} \alpha$ is the action of γ'. Let J denote an admissible almost complex structure. We may assume that $J|_{\gamma'} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ with respect to the ordered basis $\{v_1 = \partial_q, v_2 = \partial_p - \frac{q}{K}\partial_t\}$.

(a) A degenerated case.

Consider the differential operator $L_0 := -J\frac{d}{d\tau}$. The spectrum of L_o is a discrete set $\text{spec}(L_0) = \{\frac{2k\pi a'}{s} \mid k \in \mathbb{Z}\}$. For each $\lambda \in \text{spec}(L_0)$ the eigenspace $V_\lambda(L_0)$ is a two dimensional vector space with an almost complex structure (hence is oriented) induced by J. A winding number (relative to the trivialization v_1, v_2) around γ is defined for each eigenvector field of L_o. Two eigenvector field have the same winding number iff they belong to the same eigenspace $V_\lambda(L_0)$. In particular, the 0-eigenspace $V_0(L_0)$ is spanned by two linearly independent eigenvector fields (i.e., v_1 and v_2) of winding number 0.

(b) Another degenerated case.

Recall (23) and consider the operator

$$L := -J(\frac{d}{d\tau} - DR) \quad \text{with } DR = \begin{bmatrix} 0 & 0 \\ -\epsilon & 0 \end{bmatrix},\ 0 < \epsilon \ll 1$$

L corresponds to the case where Type $\mathfrak{T}$ Reeb orbits appear in S^1-families. L is a small deformation of L_0 in **(a)** above. We may think that L is included in a smooth homotopy L_t, $t \in [0, 1]$ so that $L_1 = L$. The winding numbers of eigenvector fields around γ are unchanged under this homotopy in the sense that two eigenvector fields of L_t belong to distinct eigenvalues of L_t if their winding numbers around γ are distinct. Thus the orientation of $V_\lambda(L_0)$ is transported along the homotopy to induce an orientation of the 2-

dimensional vector space spanned by linearly independent eigenvector fields of L with the same winding number around γ.

We focus on the eigenvector fields of L wit winding number 0. An easy computation shows that $v_1 = \partial_q$ and $v_2 = \partial_t - \frac{q}{K}\partial_t$ are two eigenvector fields with winding number 0, $\mathrm{Span}(v_1) = V_\epsilon(L)$, $\mathrm{Span}(v_2) = V_0(L)$. Note that

$$V_\epsilon(L) \oplus V_0(L) \text{ is homotopic to } V_0(L_0).$$

So up to homotopy the perturbation term DR has the effect of splitting $V_0(L_0)$ into a direct sum of a 1-dimensional vector space with positive eigenvalue and a 1-dimensional vector space with eigenvalue 0.

Recall that a further deformation will turn the S^1-families of Reeb orbits into pairs $e_{n/m}, h_{n/m}$. From the discussion following (23) we see that $V_\epsilon(L)$ will deform to $V_{\epsilon'}(L_\gamma)$ for some $\epsilon' > 0$ for $\gamma = e_{n/m}, h_{n/m}$ and h^m. On the other hand, $V_0(L)$ will deform to $V_{\delta_+}(L_\gamma)$ for some $\delta_+ > 0$ if $\gamma = e_{n/m}$; to $V_{\delta_-}(L_\gamma)$ for some $\delta_- < 0$ if $\gamma = h_{n/m}$ of h^m. This is summarized in **(c)**,**(d)** below.

(c) The elliptic case.

If γ is elliptic then $V_-(L_\gamma)$ is homotopic to $V_-(L_0)$ (and hence $V_-(L)$), $V_+(L_\gamma)$ is homotopic to $V_0(L_0) \oplus V_+(L_0)$. So $V_\pm(L_\gamma)$ are endowed with induced orientations as described earlier.

(d) The hyperbolic case.

Now suppose that γ is hyperbolic. Then $V_0(L_0)$ is deformed to a direct sum of two 1-dimensional eigenspaces

$$V_{\epsilon_+}(L_\gamma) = \mathrm{Span}(\tilde{v}_1) \text{ and } V_{\epsilon_-}(L_\gamma) = \mathrm{Span}(\tilde{v}_2)$$

of L_γ, where ϵ_+ (resp. ϵ_-) is a the smallest positive eigenvalue (resp. the largest negative eigenvalue) of L_γ; $\tilde{v}_k$ is homotopic to v_k for $k = 1, 2$. Moreover,

$$V_-(L_\gamma) = V_{<\epsilon_-}(L_\gamma) \oplus V_{\epsilon_-}(L_\gamma), \quad V_+(L_\gamma) = V_{\epsilon_+}(L_\gamma) \oplus V_{>\epsilon_+}(L_\gamma),$$

where $V_{<\epsilon_-}(L_\gamma)$ and $V_{>\epsilon_+}(L_\gamma)$ are with orientation induced from those of $V_-(L_0)$ and $V_+(L_0)$ respectively via deformation as before. Though $v_1 \wedge v_2 \sim \tilde{v}_1 \wedge \tilde{v}_2$ induces a complex orientation for $V_{\epsilon_+}(L_\gamma) \oplus V_{\epsilon_-}(L_\gamma) \cong \mathbb{C}$, in order to determine an orientation for each of $V_\pm(L_\gamma)$ we still need to choose an orientation for each of $V_{\epsilon_\pm}(L_\gamma)$ so that the combined orientation is consistent with the complex orientation. There are two choices:

1. either $o(V_{\epsilon_+}(L_\gamma)) = o(\tilde{v}_1)$ and $o(V_{\epsilon_-}(L_\gamma)) = o(\tilde{v}_2)$, or
2. $o(V_{\epsilon_+}(L_\gamma)) = o(-\tilde{v}_1)$ and $o(V_{\epsilon_-}(L_\gamma)) = o(-\tilde{v}_2)$.

It can be seen later that the vanishing of ∂^2 and, up to an isomorphism the resulting cylindrical contact homology are independent of the choices of $o(V_{\epsilon_\pm}(L_\gamma))$. Here we just make a choice for all $\gamma = h_{n/m}, h^m$ but we do not specify the choices as it will not hinder the computation of $HC(M, \xi)$.

Let ∂^* denote the dual operator of ∂ the boundary operator of cylindrical contact complex so that $\langle \partial\gamma_+, \gamma_- \rangle = \langle \gamma_+, \partial^*\gamma_- \rangle$.

Lemma 4.14. *We have for $m \geq 2$*

$$\partial^* h^{m-1} = \frac{1}{\sigma}(c_- e_{\sigma(m-1)/m} + c_+ e_{\sigma/m}).$$

For $m \geq 2$ and $0 < n < (m-1)\sigma$,

$$\partial^* h_{n/(m-1)} = c_- \cdot \frac{\kappa_{n,m-1}}{n} e_{n/m} + c_+ \cdot \frac{\kappa_{n,m-1}}{\sigma m - n - \sigma} e_{(n+\sigma)/m}$$

where $c_\pm \in \{-1, 1\}$ and $c_- c_+ = -1$.

Proof. The only part that needs to be verified is the equality $c_- c_+ = -1$. We will prove the second formula. The proof for h^{m-1} is similar (using $h^{m-1} = h_{0/(m-1)} = h_{\sigma(m-1)/(m-1)}$) and will be omitted.

Apply the action estimate Lemma 4.13 form Section 4.4 to the unperturbed case at first, i.e., the case of which Type $\mathfrak{T}$ Reeb orbits come in S^1-families indexed by (n, m). Let $\Upsilon_{n/m}$ denote the S^1-family of Reeb orbits indexed by (n, m) and let $\mathcal{M}(\Upsilon_{n/(m-1)}, \Upsilon_{n'/m})$, $n' = n$ or $n + \sigma$, denote the moduli of pseudoholomorphic cylinders that converge to some element $\gamma_{n/(m-1)} \in \Upsilon_{n/(m-1)}$ at $-\infty$, and to some element $\gamma'_{n'/m} \in \Upsilon_{n'/m}$ at ∞.

Let $C_M \subset$ be the image in M of some $\tilde{u} = (a, u) \in \mathcal{M}(\Upsilon_{n/(m-1)}, \Upsilon_{n'/m})$.

Assume at first that $n' = n$. Then $\frac{n}{m-1} > \frac{n'}{m}$, $(n', m) = (0, 1) + (n, m-1)$. $C_M \cap \Gamma = \emptyset$. C_M is tangent to the vector field $v_1 = \partial_q$ along the boundary $\gamma_{n/(m-1)}$. View C_M as a homotopy of $\gamma_{n/(m-1)}$. Since C_M is the image of a holomorphic curve, the integral of α along $C_M \cap T^2_q$ is increasing as q increases from q_0 with $\gamma_{n/(m-1)} \in T^2_{q_0}$ to q_1 with $\gamma'_{n/m} \in \Upsilon_{n/m}$. Hence $u_s(-\infty, \cdot)$ is a *positive* multiple of $v_1 = \partial_q$. The last property remains true after perturbation. I.e., if $\tilde{u} = (a, u) \in \mathcal{M}(h_{n/(m-1)}, e_{n/m})$ then $u_s(-\infty, \cdot)$ is a positive multiple of $\tilde{v}_1$. The flow u_s, $s \in \mathbb{R}$ induces an orientation on $V_-(e_{n/m})$ that is compatible with the orientation

$$o_- := o(\tilde{v}_1) \oplus o(V_-(h_{n/(m-1)})). \tag{35}$$

On the other hand, if $n' = n + \sigma$ then $\frac{n'}{m} > \frac{n}{m-1}$ (recall that $n < (m-1)\sigma$), $(n', m) = (\sigma, 1) + (n, m-1)$. C_M has to cross Γ_ϕ. By applying argument similar to the one in the case $n' = n$ we find that if $\tilde{u} = (a, u) \in \mathcal{M}(h_{n/(m-1)}, e_{(n+\sigma)/m})$ then $u_s(-\infty, \cdot)$ is a *negative* multiple of $\tilde{v}_1$. The flow u_s, $s \in \mathbb{R}$ induces an orientation on $V_-(e_{(n+\sigma)/m})$ that is compatible with the orientation

$$o_+ := o(-\tilde{v}_1) \oplus o(V_-(h_{n/(m-1)}). \tag{36}$$

It is easy to see that *exactly one* of the following two equalities holds true:

$$o_- = o(V_-(e_{n/m})); \quad o_+ = o(V_-(e_{(n+\sigma)/m})).$$

So we have $c_- c_+ = -1$. □

Lemma 4.14, Proposition 4.4, Lemma 4.12 and (8) lead to the following

Lemma 4.15. *The boundary operator ∂ of the contact complex of (M, ξ) satisfies the following equations.*

1. $\partial h^m = 0 = \partial B^m$ *for* $m \in \mathbb{N}$;
2. $\partial h_{n/m} = 0$ *for* $n, m \in \mathbb{N}$, $0 < n < \sigma m$;
3. $\partial e_{n/1} = 0$ *for* $n \in N$, $0 < n < \sigma$;
4. $\partial e_{n/m} = c_- \cdot \frac{n}{\kappa_{n,m-1}} h_{n/(m-1)}$ *for* $0 < n < \sigma$ *and* $m \geq 2$;
5. $\partial e_{n/m} = c_+ \cdot \frac{\sigma m - n}{\kappa_{n-\sigma,m-1}} h_{(n-\sigma)/(m-1)}$ *for* $\sigma(m-1) < n < \sigma m$ *and* $m \geq 2$;
6. $\partial e_{n/m} = c_- \cdot \frac{n}{\kappa_{n,m-1}} h_{n/(m-1)} + c_+ \cdot \frac{\sigma m - n}{\kappa_{n-\sigma,m-1}} h_{(n-\sigma)/(m-1)}$ *for* $\sigma \leq n \leq \sigma(m-1)$ *and* $m \geq 2$,

where $c_-, c_+ \in \{-1, 1\}$ *and* $c_- c_+ = -1$.

Note that the equation $\partial h_{n/m} = 0$ can be obtained by a Mores-Bott argument.

Corollary 4.2. $\partial^2 = 0$. *Hence the cylindrical contact homology $HC(M, \xi)$ is defined. Moreover, by Corollary 4.1 and Theorem 2.1, $HC(M, \xi)$ is independent of the choice of (α, J), and is an invariant of the isotopy class of ξ.*

4.6 Computing $HC(M,\xi)$

Let $\mathcal{H}$ denote the subcomplex generated by all hyperbolic Reeb orbits, and $\mathcal{E}$ the subcomplex generated by all elliptic Reeb orbits (including no B^m).

The lemma below follows from Lemma 4.15.

Lemma 4.16. *$\mathcal{H} \subset \ker \partial$. In particular every hyperbolic Reeb orbit is a generator of $HC(M,\xi)$. Also, $\mathcal{E} \cap im\partial = 0$.*

Lemma 4.17. *$[h_{n/m}] = 0 \in HC(M,\xi)$ for all $0 < n < \sigma m$ with n not divisible by σ.*

Proof. Let $0 < i < \sigma$. From Lemma 4.15 we know that for $k = 2, ..., m-1$, $h_{(k\sigma-i)/m}$ is homologous (as an element of $HC(M,\xi)$) to a nonzero rational multiple of $h_{(\sigma-i)/m}$, and the latter is a nonzero rational multiple of $\partial e_{(\sigma-i)/m}$. This completes the proof. $\square$

Lemma 4.18. *For $m \geq 1$ and $0 < k < m$, $[h_{k\sigma/m}]$ is a nonzero rational multiple of $[h^m]$ in $HC(M,\xi)$. Also, $0 \neq [h] \in HC(M,\xi)$.*

Proof. The first statement follows from Lemma 4.15. Use the equations

$$[h_{(k-1)\sigma/m}] = \frac{k\kappa_{(k-1)\sigma,m}}{(m+1-k)\kappa_{k\sigma,m}}[h_{k\sigma/m}], \quad k = 1, 2, ..., m,$$

we get

$$\begin{aligned}[h^m] = [h_{0/m}] &= \Big(\prod_{k=1}^{m} \frac{k}{m+1-k} \cdot \prod_{k=1}^{m} \frac{\kappa_{(k-1)\sigma,m}}{\kappa_{k\sigma,m}}\Big)[h_{m\sigma/m}] \\ &= 1 \cdot 1 \cdot [h^m]\end{aligned}$$

There are no other boundary relations about h^m, so $0 \neq [h] \in HC(M,\xi)$. $\square$

Definition 4.2. *For $m \in \mathbb{N}_{\geq 2}$ define*

$$E_{0,m} := e_{\sigma/m} + \sum_{k=2}^{m-1} \Big(\prod_{j=1}^{k-1} \frac{j}{(m-j-1)}\Big) e_{k\sigma/m}.$$

Also for $i, m \in \mathbb{N}$ with $0 < i < \sigma$ define

$$E_{i,m} := \begin{cases} e_{i/1} & \text{if } m = 1, \\ e_{(\sigma-i)/m} + \sum_{k=2}^{m} \Big(\prod_{j=1}^{k-1} \frac{j\sigma-i}{(m-j-1)\sigma+i}\Big) e_{(k\sigma-i)/m} & \text{if } m \geq 2. \end{cases}$$

Lemma 4.19. *$\partial E_{i,m} = 0$ for all $(i,m) \in \mathbb{Z}_\sigma \times \mathbb{N} \setminus \{(0,1)\}$.*

Proof. Apply Lemma 4.15. $\square$

Lemma 4.20. *Let E' be a finite linear combination of $e_{n/m}$'s with $\mathbb{Q}$-coefficients. Suppose that $\partial E' = 0$. Then E is a finite linear combination of $E_{i,m}$'s.*

Proof. Write

$$E' = \sum_{n,m} c_{n,m} e_{n/m} = \sum_{i,m} E'_{i,m}, \qquad (i,m) \in \mathbb{Z}_\sigma \times \mathbb{N} \setminus \{(0,1)\},$$

where $E'_{i,m}$ consists of all $e_{n/m}$ terms of E' with $n \equiv i \mod \sigma$. It is easy to see that

$$\partial E' = 0 \text{ iff } \partial E'_{i,m} = 0 \quad \forall i, m.$$

Note that when $m = 1$ and $0 < i < \sigma$, each $E'_{i,1}$ is a rational multiple of $e_{i/1} = E_{i/1}$ (hence satisfies $\partial E'_{i,1} = 0$), so we only need to consider the case $m \geq 2$.

Fix i, m with $m \geq 2$ and assume that $\partial E'_{i,m} = 0$. Let n_o be the smallest integer such that $n_o \equiv i \mod \sigma$ and $c_{n_o,m} \neq 0$.

Claim: $n_o \leq \sigma$.

First assume instead that $n_o > \sigma(m-1)$, then $\partial E'_{i,m} = \partial(c_{n_o/m} e_{n_o/m}) \neq 0$ by Lemma 4.15, contradicting with $\partial E'_{i,m} = 0$. So $n_o \leq \sigma(m-1)$.

Now suppose that $\sigma < n_o \leq \sigma(m-1)$, then

$$\begin{aligned} 0 = \partial E'_{i,m} &= \partial c_{n_o,m} e_{n_o/m} + \partial\Big(\sum_{k=1}^{\lfloor m - \frac{n_o}{\sigma} \rfloor} c_{(n_o+k\sigma),m} e_{(n_o+k\sigma)/m} \Big) \\ &= \Big(c_0 h_{(n_o-\sigma)/(m-1)} + c_1 h_{n_o/(m-1)} \Big) + \Big(\text{no } h_{(n_o-\sigma)/(m-1)} \text{ term here } \Big) \\ &\neq 0 \quad \text{ because } c_0 \neq 0. \quad \text{A contradiction again!} \end{aligned}$$

So $n_o \leq \sigma$.

Suppose that $n_o < \sigma$, then $\partial e_{n_o/m}$ is a nonzero rational multiple of $h_{n_o/(m-1)}$ which can be cancelled only by adding a certain rational multiple of $\partial e_{(n_o+\sigma)/m}$, which again generates a nonzero rational multiple of $h_{(n_o+\sigma)/m}$ that can be cancelled only by adding a certain rational multiple of $\partial e_{(n_o+2\sigma)/m}$, and so on so forth. This process actually yields $E_{m,i}$ as defined in Definition 4.2. So $E'_{i,m} = c_{i,m} E_{i,m}$ for $i = 1, ..., \sigma - 1$. Similar arguments also apply to the case $n_o = \sigma$ and we have $E'_{0,m} = c_{\sigma,m} E_{0,m}$. This completes the proof. $\square$

We have the following

Lemma 4.21. *$HC(M, \xi)$ is freely generated by $[h^m]$, $m \geq 1$, and by $E_{i,m}$, $(i, m) \in \mathbb{Z}_\sigma \times \mathbb{N} \setminus \{(0, 1)\}$. The homologously trivial (as elements of $H_1(M, \mathbb{Z})$) generators are $[h^m]$, $m \geq 1$, and $E_{0,m}$ with $m \geq 2$.*

Note that $[h^m] = 0 = [E_{0,m}] \in H_1(M, \mathbb{Z})$ (they are contractible actually), so their $\bar{\mu}$-indexes are defined. Recall from Lemma 4.6 that $\bar{\mu}(h^m) = 2m - 1$ for $m \in \mathbb{N}$. Also by (26) we have $\bar{\mu}(E_{0,m}) = \bar{\mu}(e_{\sigma/m}) = 2m - 2$ for $m \in \mathbb{N}_{\geq 2}$. This completes the proof of Theorem 1.2.

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